An Improved Parameterized Algorithm for Treewidth

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UNIVERSITY OF BERGEN

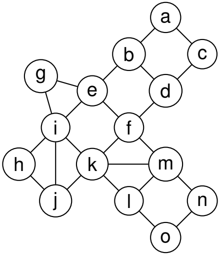
joint work with
Daniel Lokshtanov[1]

[1] University of California Santa Barbara

UCSB CS Theory Colloquium

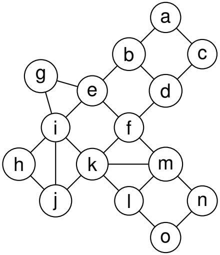
9 November 2022

Treewidth



Graph G

Treewidth



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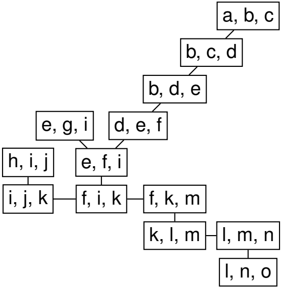
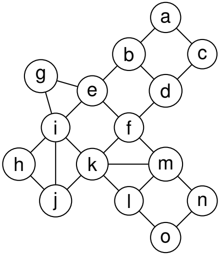


A tree decomposition of G

Treewidth



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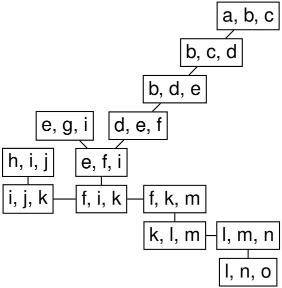

A tree decomposition of G

1. Every vertex should be in a bag

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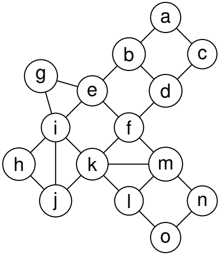

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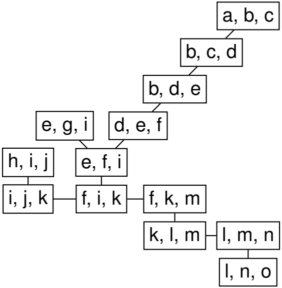

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1. Every vertex should be in a bag
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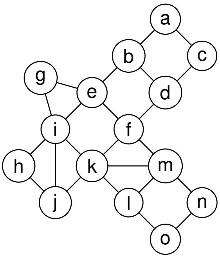

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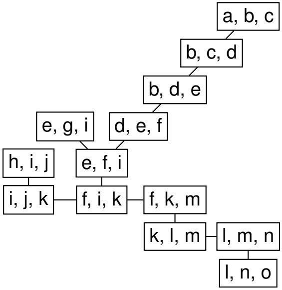

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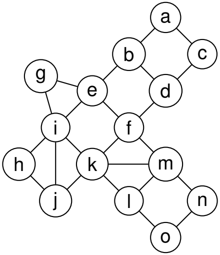

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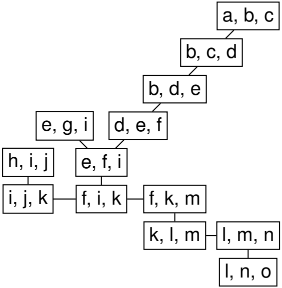

A tree decomposition of G

1. Every vertex should be in a bag
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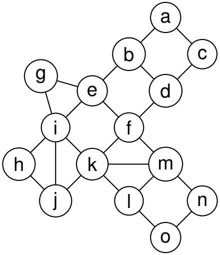
4. Width = maximum bag size -1

Treewidth



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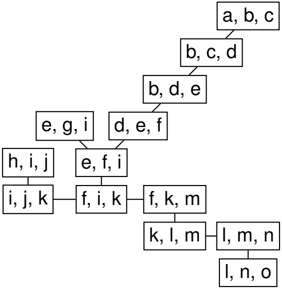

A tree decomposition of G
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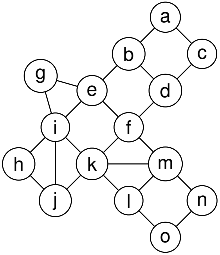

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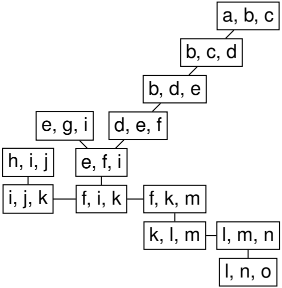

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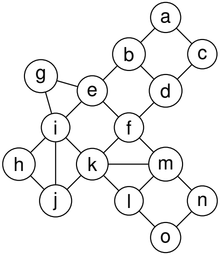

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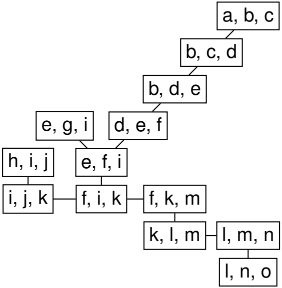

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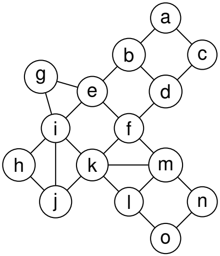

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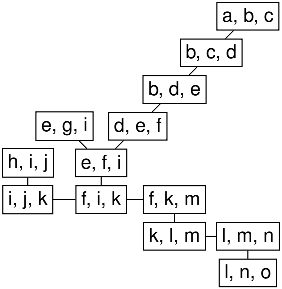

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[Bertele & Brioschi'72, Halin'76, Robertson & Seymour'84]

Treewidth of graphs

Some graphs of small treewidth:

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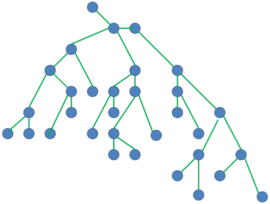
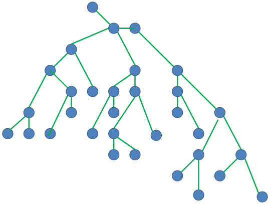


Trees (tw ≤ 1)

Treewidth of graphs

Some graphs of small treewidth:



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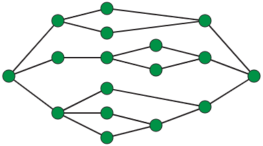

Series-parallel (tw $\leq$ 2)

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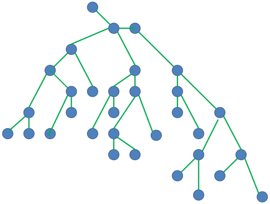

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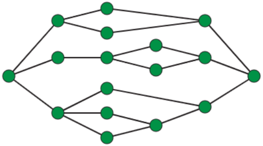

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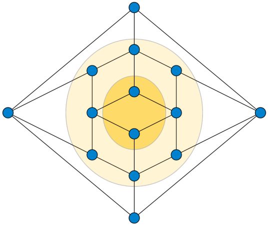

k-outerplanar (tw $\leq 3k-1$)

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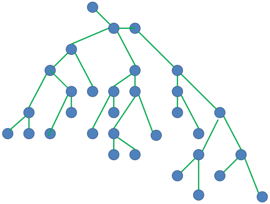

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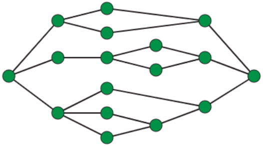

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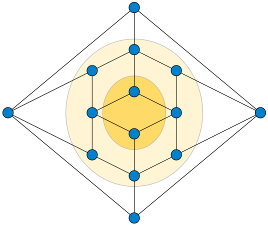

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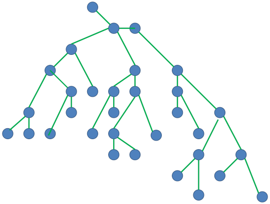

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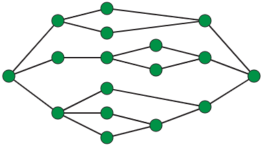

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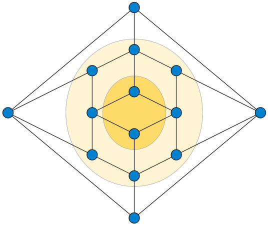

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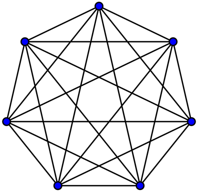

Clique (tw $= n - 1$)

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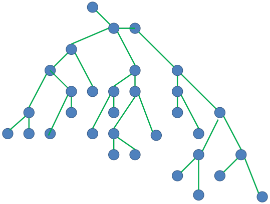

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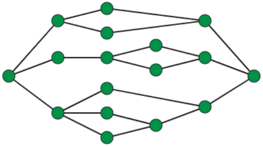

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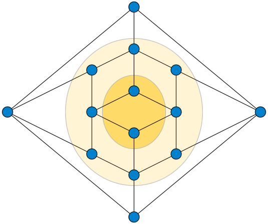

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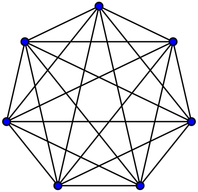

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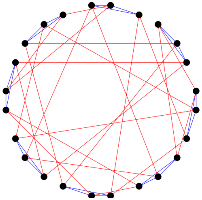

Expanders (tw $= \Theta(n)$)

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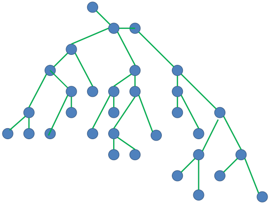

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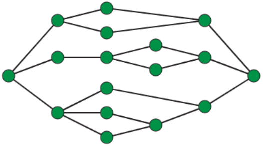

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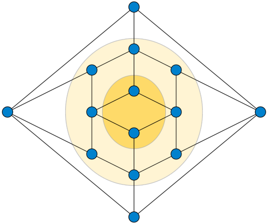

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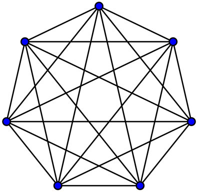

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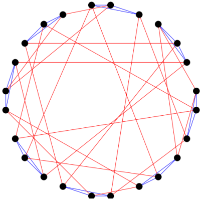

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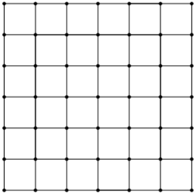

$n \times n$-grid (tw $= n$)

Why treewidth: Graph theory

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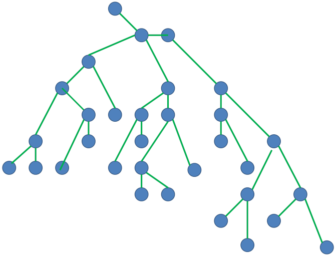
Dichotomy: Either small treewidth, or large grid inside



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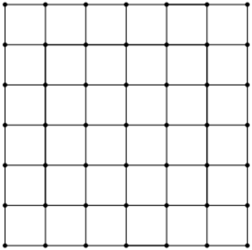

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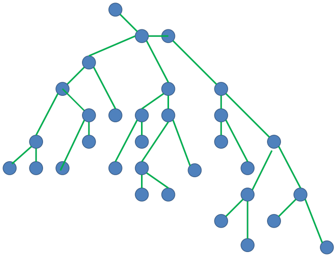

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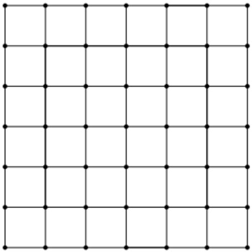
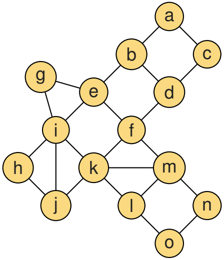
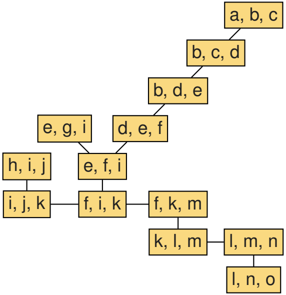


Grid minor theorem (Robertson & Seymour'86)

Exists function $f(k)$ s.t. any graph with tw$\geq f(k)$ has a $k \times k$-grid minor.

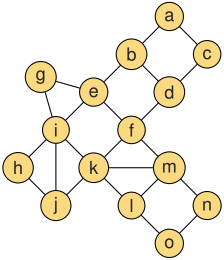
Why treewidth: Algorithms

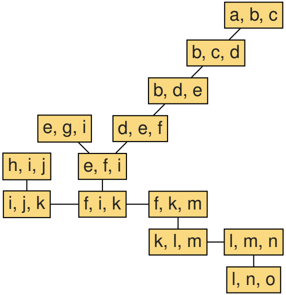




Why treewidth: Algorithms

- Algorithms on trees generalize to algorithms on graphs of small treewidth





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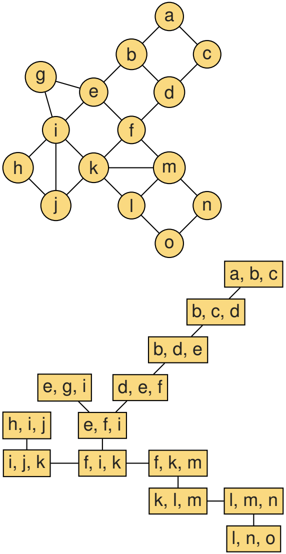
- Given a graph with a tree decomposition of width k:



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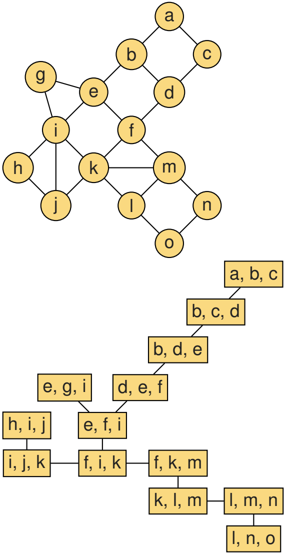

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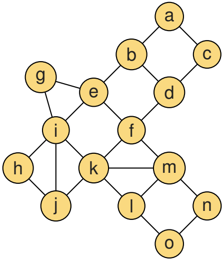

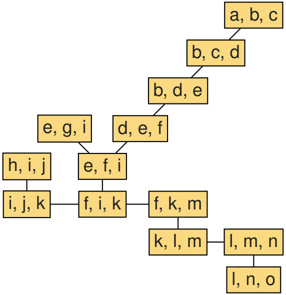

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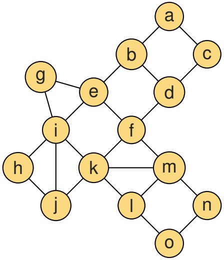

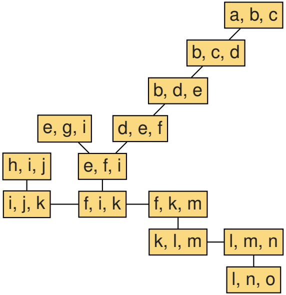

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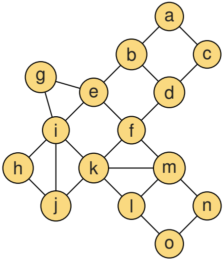

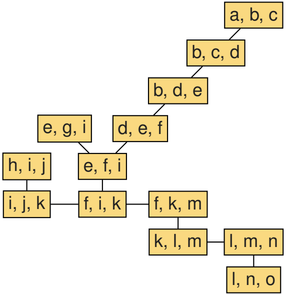

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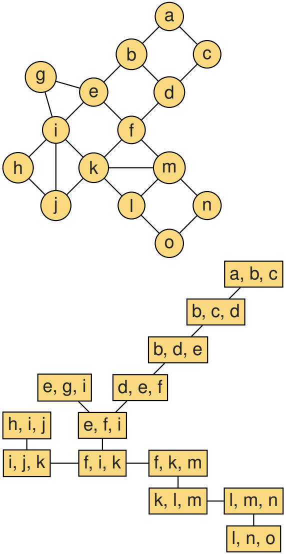
- Need to compute the tree decomposition first!



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Computing treewidth (and the tree decomposition)

- NP-complete [Arnborg, Corneil, Proskurowski '87]
- $\mathcal{O}(n^{k+2})$ time [Arnborg, Corneil, Proskurowski '87]
- $f(k) \cdot n^2$ time, non-constructive [Robertson & Seymour'86]

Computing treewidth (and the tree decomposition)

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Theorem (This work)

There is a $2^{\mathcal{O}(k^2)} n^4$ time algorithm for treewidth.

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There is a $k^{\mathcal{O}(k/\varepsilon)} n^4$ time $(1 + \varepsilon)$-approximation algorithm for treewidth.

Outline

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1. How to improve a tree decomposition

Suffices to solve the *Subset treewidth* problem

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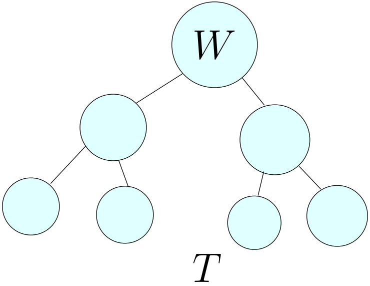
2. Solving the subset treewidth problem

Algorithms for subset treewidth that then imply algorithms for treewidth

How to improve a tree decomposition

Setting

Suppose we have a tree decomposition T whose largest bag is W



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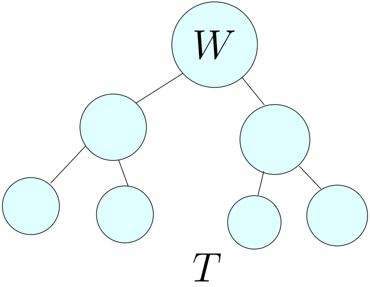

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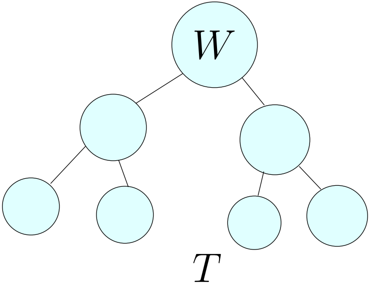

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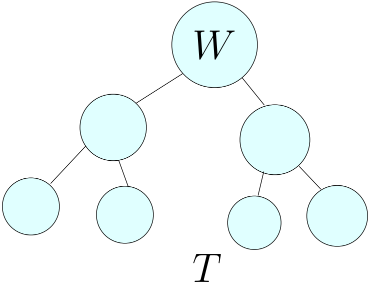

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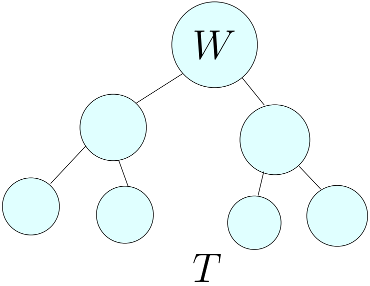

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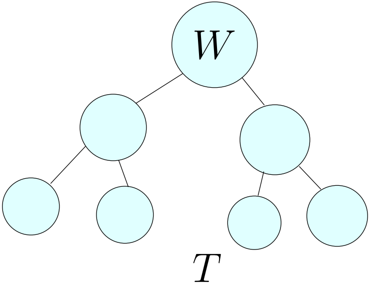
(assume to start with width $\mathcal{O}(\text{tw}(G))$ decomposition)



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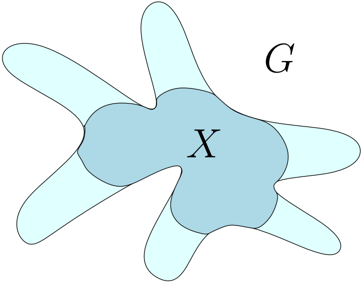

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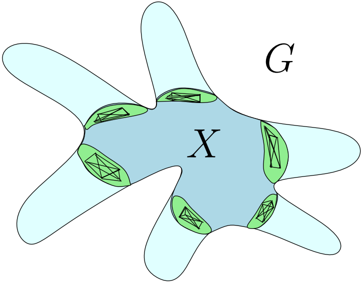

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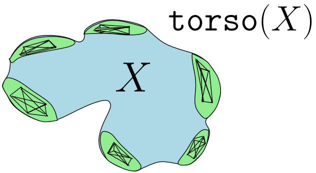

- Make neighborhoods of components of $G \setminus X$ into cliques
- Delete $V(G) \setminus X$

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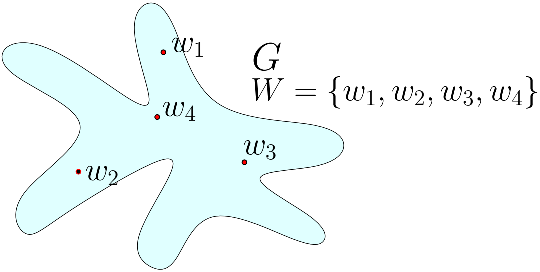

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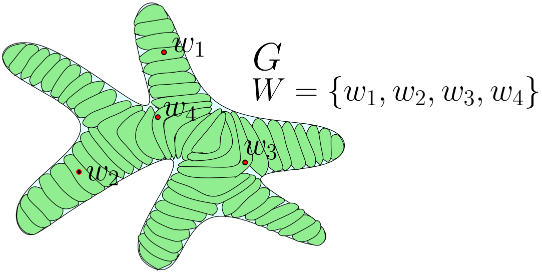

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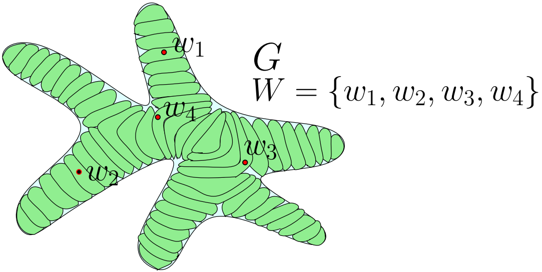

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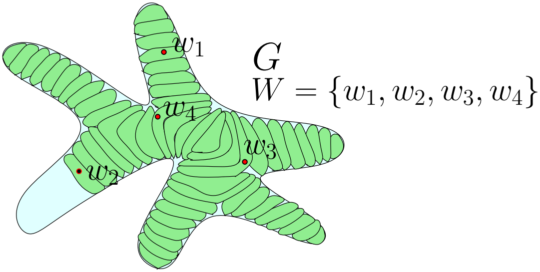

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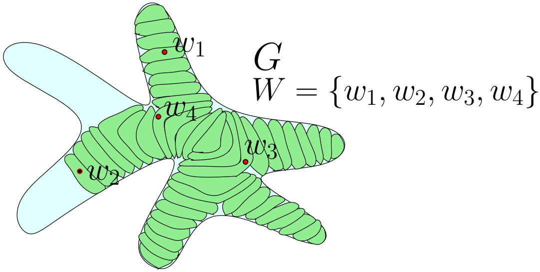

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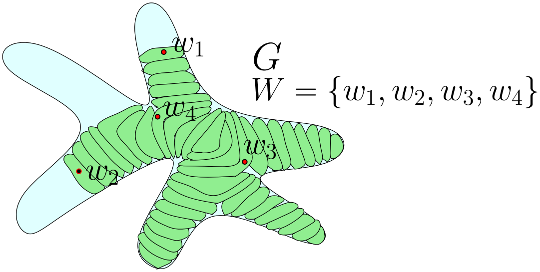

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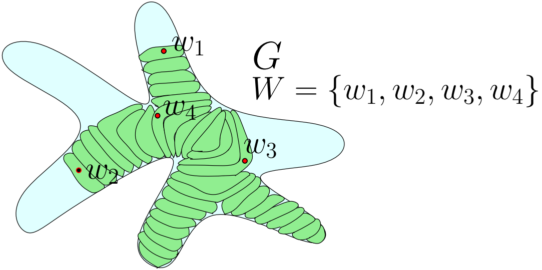

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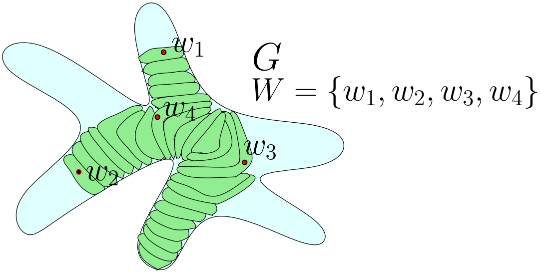

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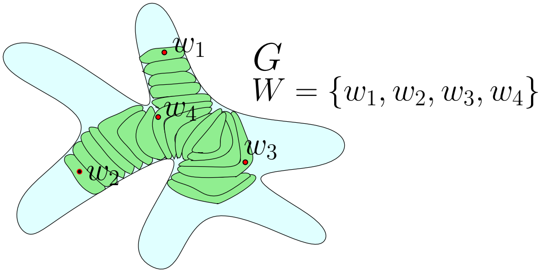

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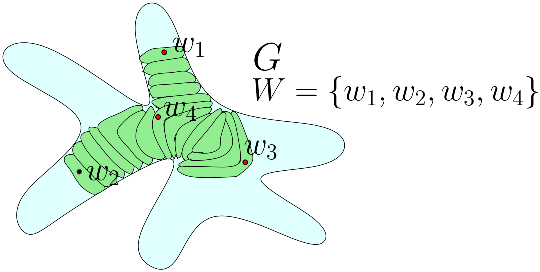

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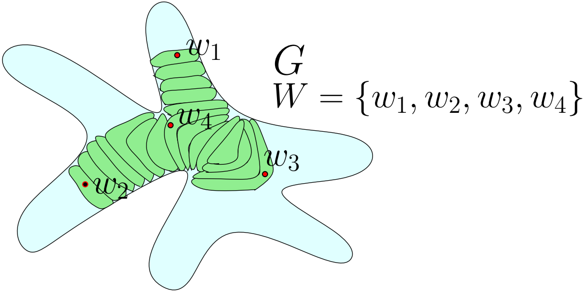

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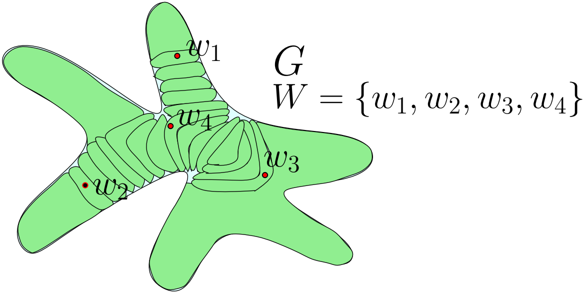
Big-leaf formulation:

- Find a tree decomposition of G whose internal bags have size $\leq |W| - 1$ and cover W, but leaf bags can be arbitrarily large



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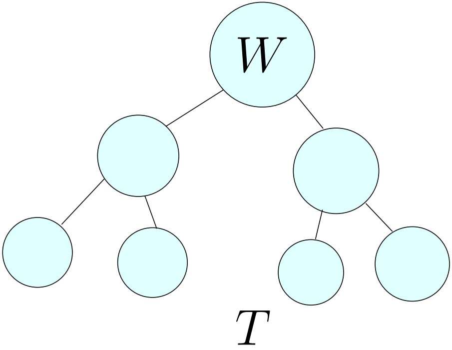

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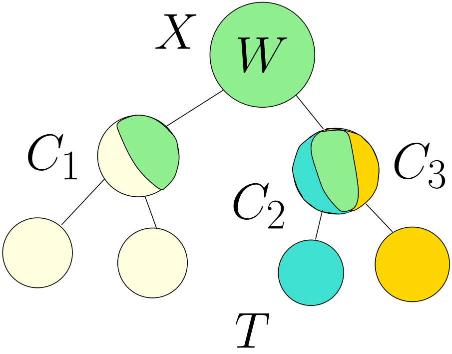

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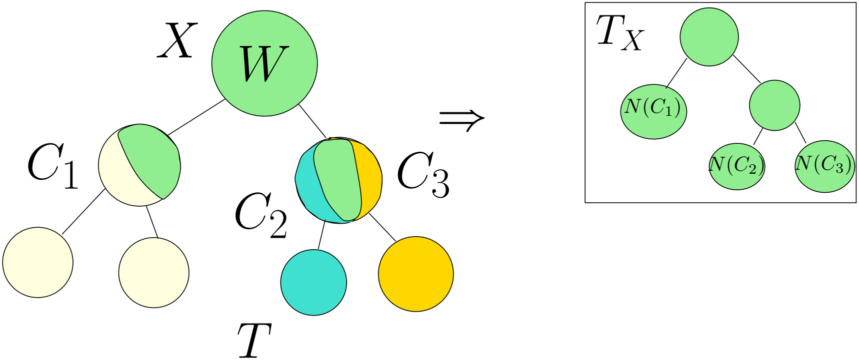

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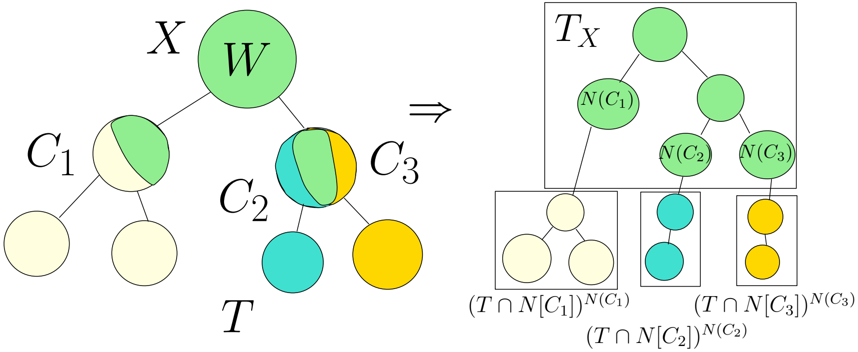

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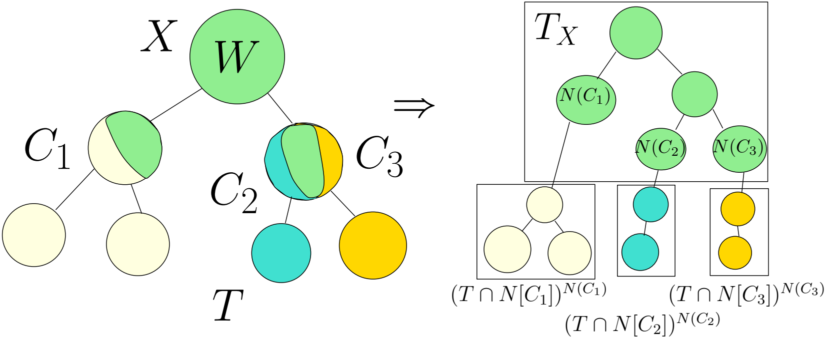

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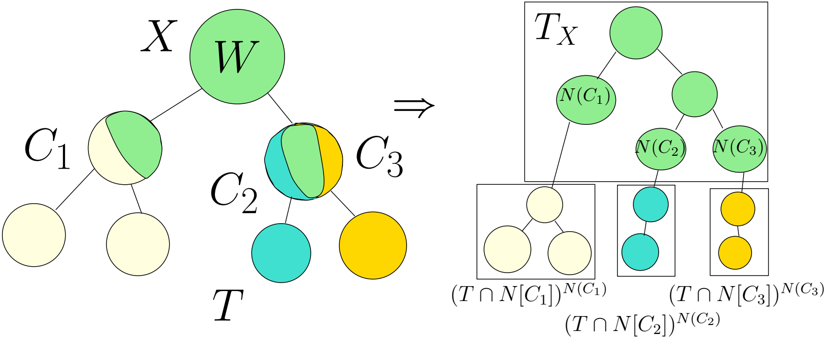

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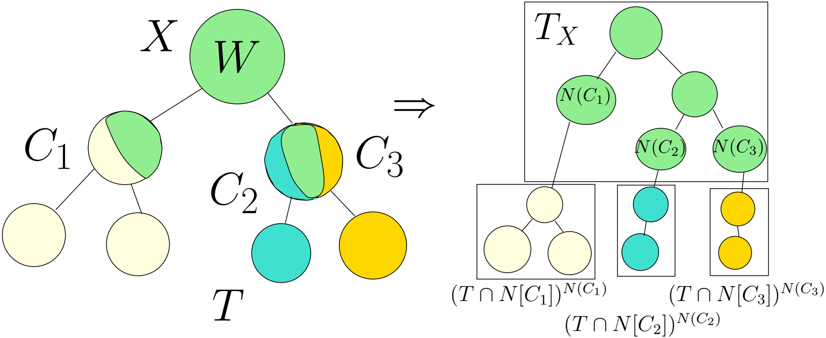

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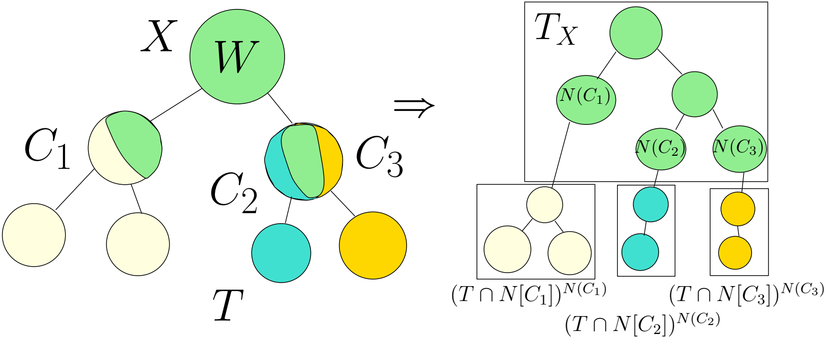

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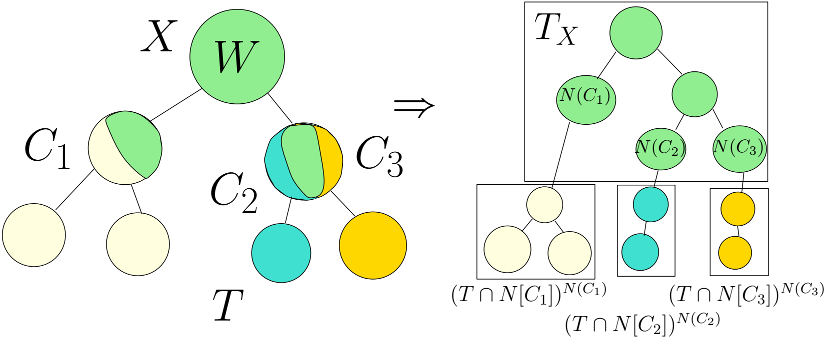

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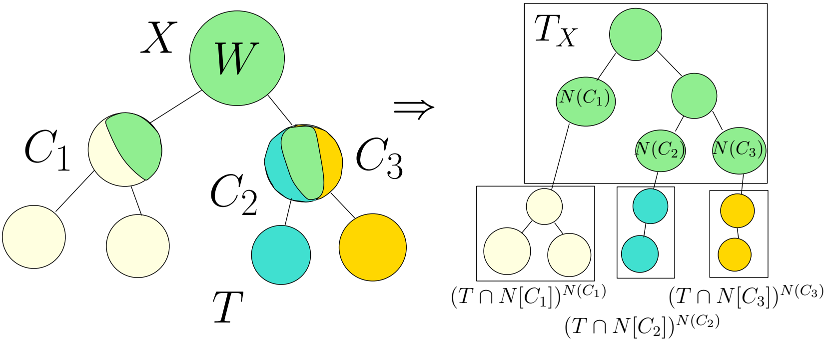

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- (actually need a bit stronger condition than linkedness for improvement)

Subset treewidth for exact algorithms

Subset treewidth for exact algorithms

SUBSET TREEWIDTH

Input: Graph G, integer k, set of vertices $W \subseteq V(G)$ with $|W| = k + 2$

Output: Set $X \subseteq V(G)$ with $W \subseteq X$ and tree decomposition of $\mathtt{torso}(X)$ of width $\leq k$ or that the treewidth of G is $> k$

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If there is an $f(k) \cdot n^{\mathcal{O}(1)}$ time algorithm for subset treewidth, then there is an $f(k) \cdot n^{\mathcal{O}(1)}$ time algorithm for treewidth with the same function f.

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$2^{\mathcal{O}(k^2)} n^2$ time algorithm for subset treewidth $\rightarrow$ $2^{\mathcal{O}(k^2)} n^4$ time algorithm for treewidth

Subset treewidth for approximation schemes

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PARTITIONED SUBSET TREEWIDTH

Input: Graph G, integer k, set of vertices $W \subseteq V(G)$ with $|W| = k+2$ that is partitioned into t cliques $W_1, \ldots, W_t$

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- Idea: Can afford to increase treewidth by εk
- Any set W can be partitioned into $t = \mathcal{O}(1/\varepsilon)$ sets $W_1, \ldots, W_t$ so that making them into cliques increases treewidth by at most $\varepsilon |W|$

Solving the subset treewidth problem

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Goal: Sketch $k^{\mathcal{O}(kt)} n^{\mathcal{O}(1)}$ time algorithm for partitioned subset treewidth

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(this is also a $k^{\mathcal{O}(k^2)} n^{\mathcal{O}(1)}$ time algorithm for subset treewidth)

Solving subset treewidth

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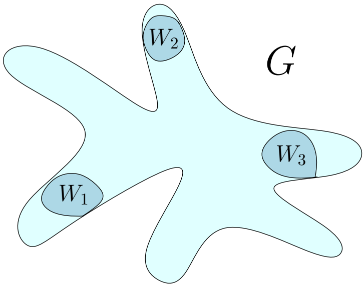

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- Input: Graph G, t terminal cliques $W_1, \ldots, W_t$, and an integer k
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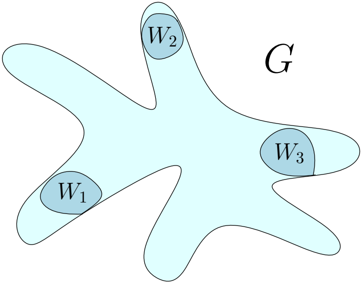

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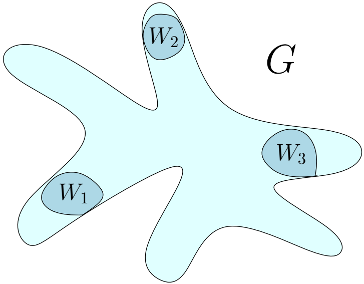

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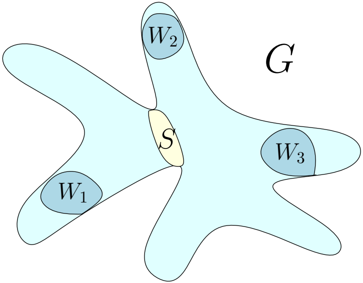

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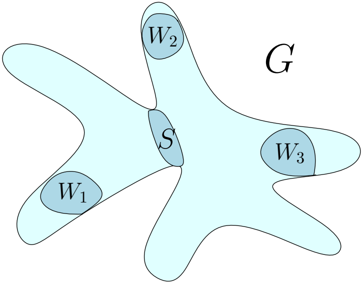
Make S into a terminal clique and solve both sides independently



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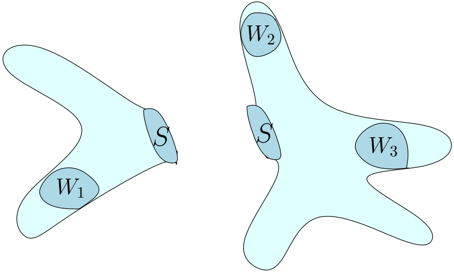
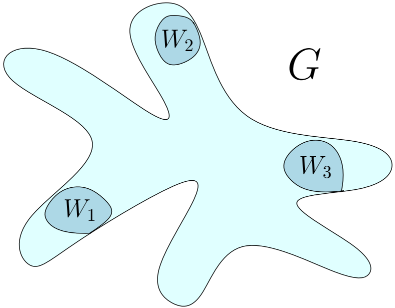


Branching for partitioned subset treewidth

- Now terminal cliques *strongly linked* into each other



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- Now terminal cliques *strongly linked* into each other
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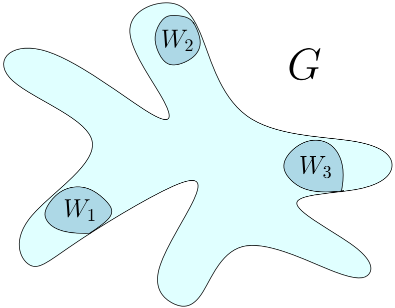

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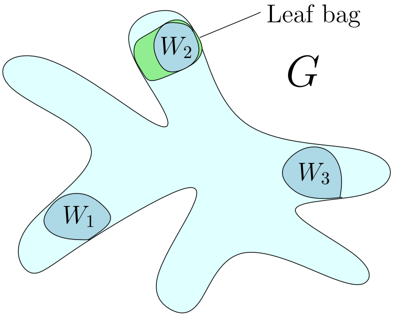

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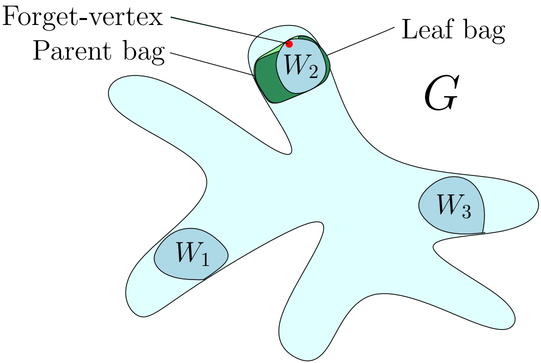

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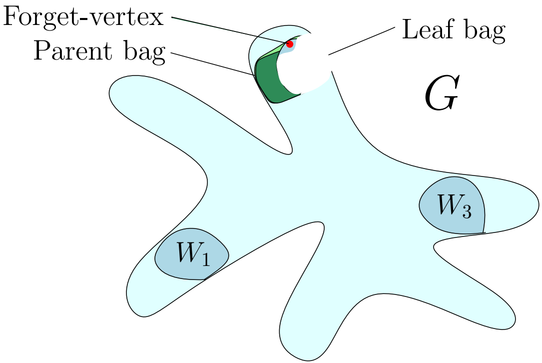

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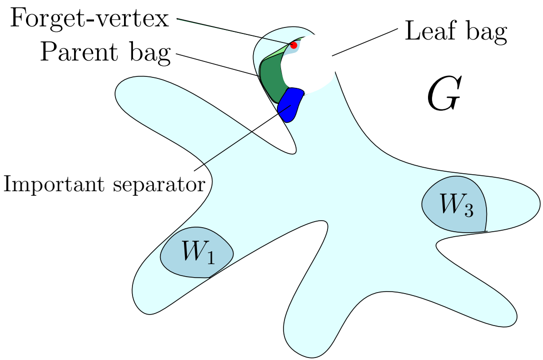

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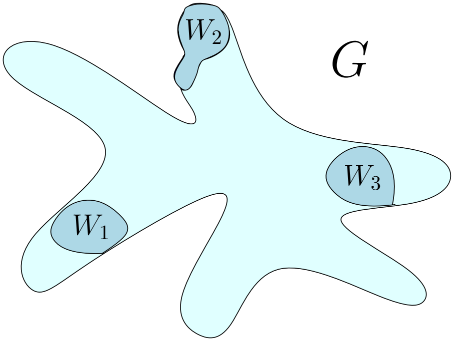
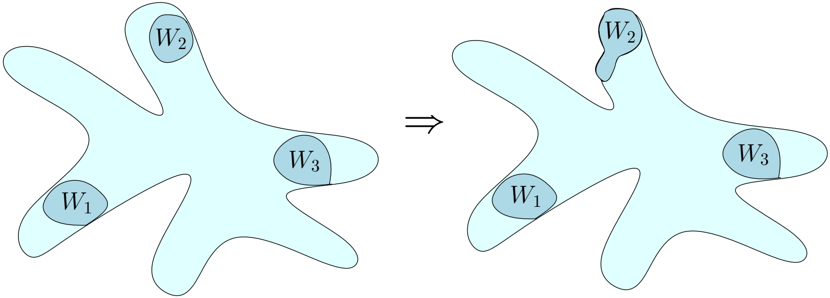


Analysis of branching



- Increased the size/flow of a leaf terminal clique by guessing a forget-vertex and an important separator

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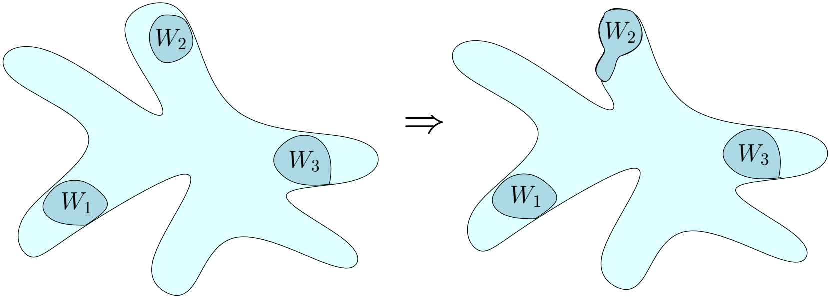

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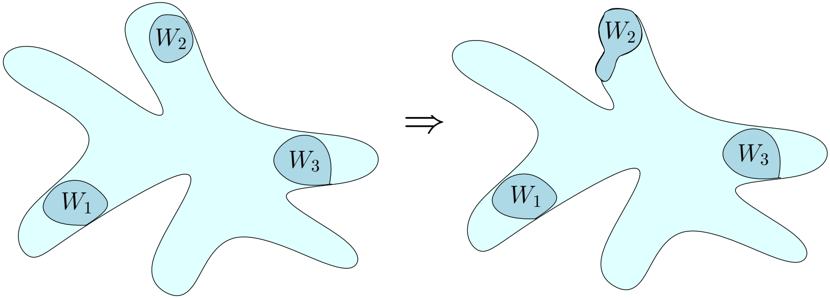

- Increased the size/flow of a leaf terminal clique by guessing a forget-vertex and an important separator
- Sum of sizes/flows of terminal cliques at most $(k+1)t$, so branching depth at most kt
- To get $k^{\mathcal{O}(kt)} n^{\mathcal{O}(1)}$ time, need also an important separator hitting set lemma

Conclusion

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- How much can the n^4 factor be optimized?

Thank you!