Linear-Time Algorithms for k-Edge-Connected Components, k-Lean Tree Decompositions, and More

Tuukka Korhonen

UNIVERSITY OF COPENHAGEN

STOC 2025

23 June 2025

Graph connectivity problems

Almost-linear-time algorithms for many problems:

- Edge connectivity (global edge min-cut), $\mathcal{O}(m \log^3 n)$ [Karger '96]
- Vertex connectivity (global vertex min-cut), $\mathcal{O}(m^{1+o(1)})$ [Li, Nanongkai, Panigrahi, Saranurak & Yingchareonthawornchai '21]
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- Deterministic!
- One graph decomposition to rule them all!

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- $\mathcal{O}(m)$ for $k = 3$ [Galil & Italiano '91] (using [Hopcroft & Tarjan '73])
- $\mathcal{O}(m)$ for $k = 4$ [Nadara,Radecki,Smulewicz&Sokołowski'21, Georgiadis,Italiano&Kosinas'21]
- $\mathcal{O}(m)$ for $k = 5$ [Kosinas '24]

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For minimum cut:

- $\mathcal{O}(k^2 m \log m)$ [Gabow '91], $\mathcal{O}(m \log m)$ [Karger '96]

k-Lean Tree Decompositions and More

Main technical result:

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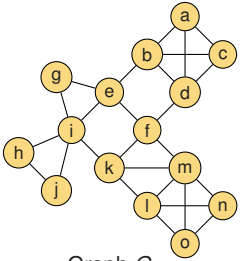
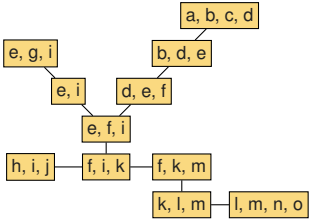
- k-Unbreakable tree decomposition in $k^{\mathcal{O}(k^2)}m$ time (with optimal unbreakability parameters)
  - Previously $k^{\mathcal{O}(k)}n^{\mathcal{O}(1)}$ [Cygan, Komosa, Lokshtanov, Pilipczuk, Pilipczuk, Saurabh, Wahlström '21]
  - and $k^{\mathcal{O}(k)}m^{1+o(1)}$ [Anand, Lee, Li, Long, Saranurak '25] (suboptimal unbreakability parameters)

k-Lean Tree Decompositions



Graph G



A 3-lean tree decomposition of G

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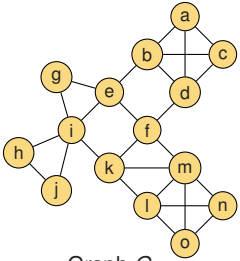

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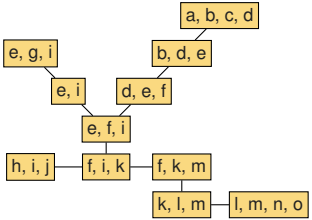

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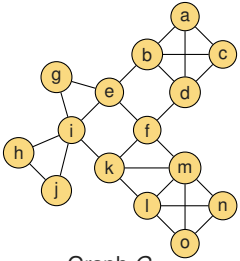

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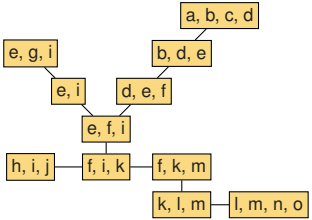

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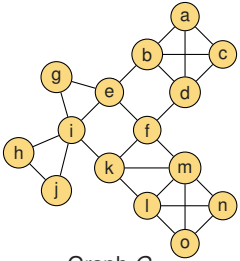

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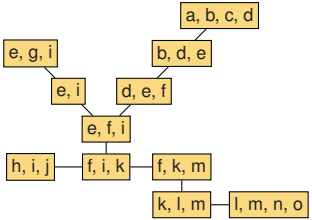

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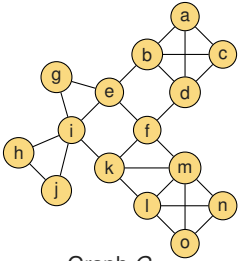

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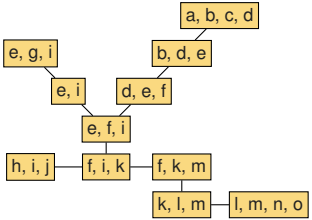

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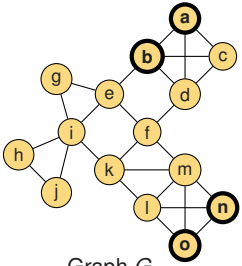

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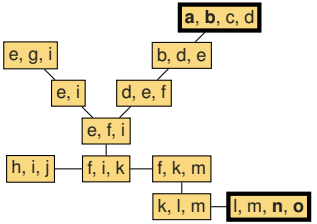

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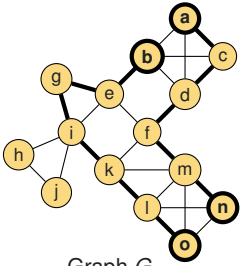

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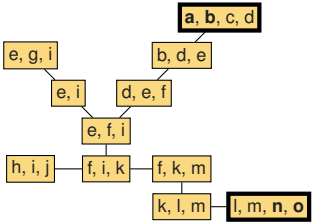

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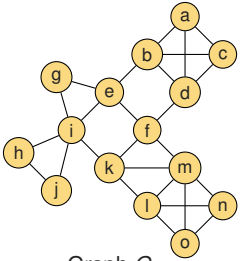

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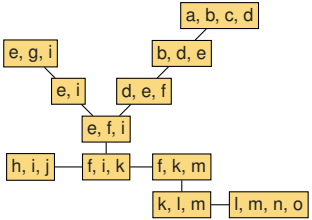

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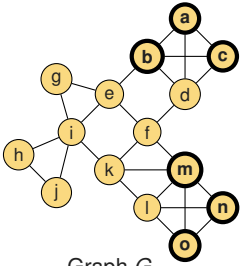

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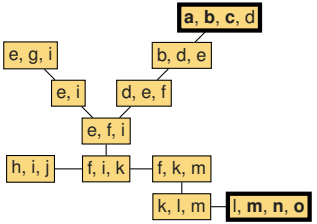

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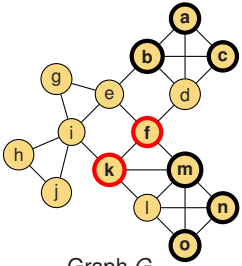

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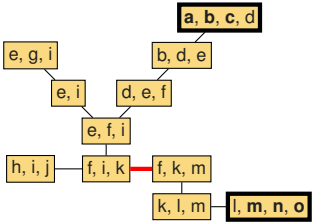

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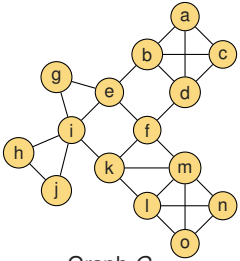

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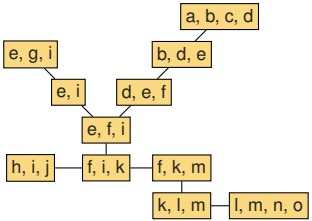

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- Defined by [Thomas '90] (for $k = \infty$), and [Carmesin, Diestel, Hamann, and Hundertmark '14]

The algorithm

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Part 1: Proof that “improver algorithm” implies the algorithm (Inspired by [Bodlaender ’96])

Part 2: The improver algorithm (Inspired by [Graph Minors X., Robertson & Seymour ’91])

Part 1: Improver algorithm implies the algorithm

Improver algorithm:

Input: A "weakly-k-lean" tree decomposition:

- Adhesion size $< 2k$
- Any two subsets $X_1, X_2 \subseteq B$ of a bag B of size $|X_1|, |X_2| \geq 2k$ can be linked by k vertex-disjoint paths

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Lemma

If there is improver algorithm with running time $f(k) \cdot m$, then there is an algorithm that in time $\text{poly}(k) \cdot f(k) \cdot m$ computes a k-lean tree decomposition.

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- Any two subsets $X_1, X_2 \subseteq B$ of a bag B of size $|X_1|, |X_2| \geq 2k$ can be linked by k vertex-disjoint paths

Output: k-lean tree decomposition

Lemma

If there is improver algorithm with running time $f(k) \cdot m$, then there is an algorithm that in time $\text{poly}(k) \cdot f(k) \cdot m$ computes a k-lean tree decomposition.

Proof idea:

- Self-reduction by recusion contracting a linear-size matching à la [Bodlaender '96]
- Sparsification by [Nagamochi-Ibaraki '92]

Part 2: The improver algorithm

Improver algorithm:

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There is an improver algorithm with running time $k^{\mathcal{O}(k^2)}m$.

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  - Separation (L, S, R), so that S is well-linked in both $G[L \cup S]$ and $G[S \cup R]$
  - Allows greedy decomposition

Conclusion

- $k^{\mathcal{O}(k^2)}m$ time algorithm for k-lean tree decomposition, implying algorithms for:
  - k-edge-connected components (long-standing open problem)
  - k-vertex connectivity
  - k-unbreakable tree decomposition
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Main techniques:

- Recursive matching contraction compression (inspired by [Bodlaender '96])
- Decomposition by doubly well-linked separations (inspired by [Graph Minors X., Robertson & Seymour '91])

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Open problems/future work:

- Improve running time to $2^{o(k^2)}m$ (even e.g. just for k-edge-connected components)
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Thank you!