An Improved Parameterized Algorithm for Treewidth

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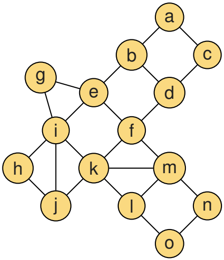
Treewidth

- Measures how close a graph is to a tree



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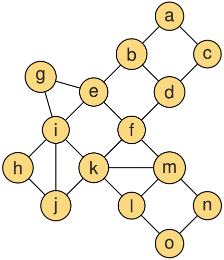

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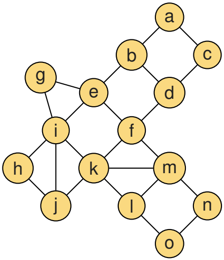

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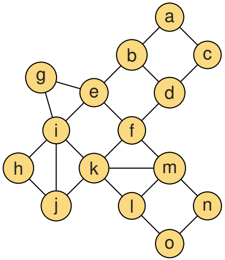

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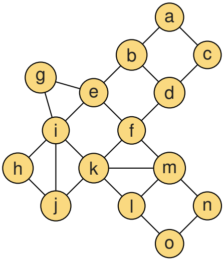

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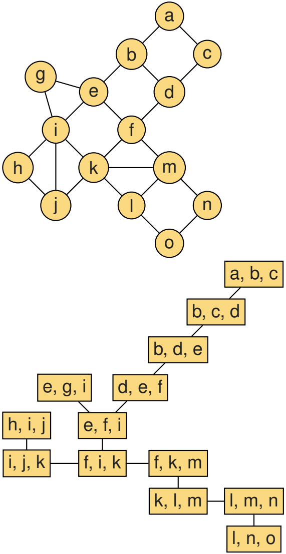

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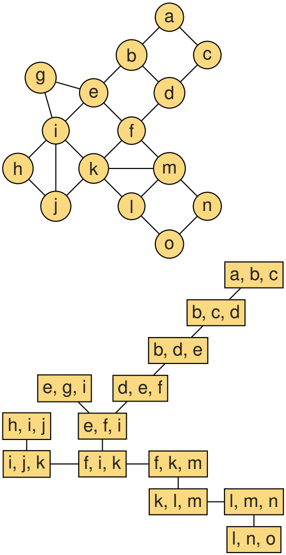

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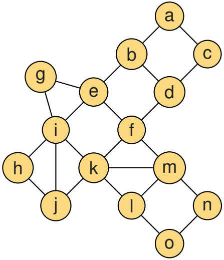

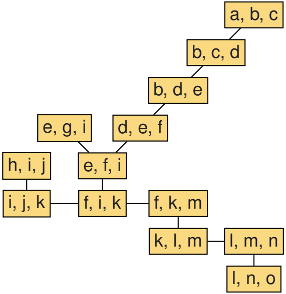

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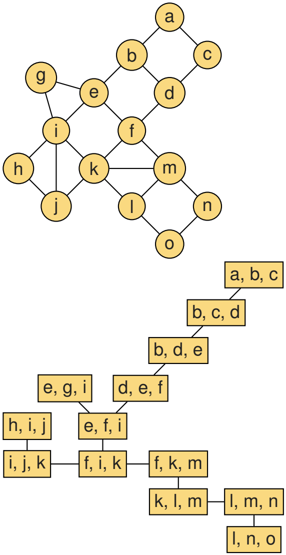

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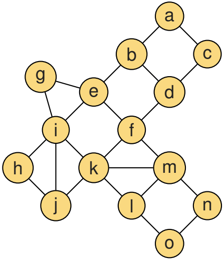

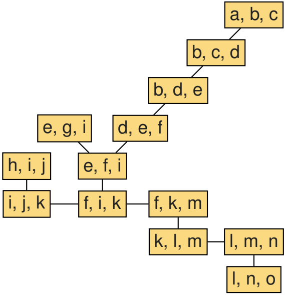

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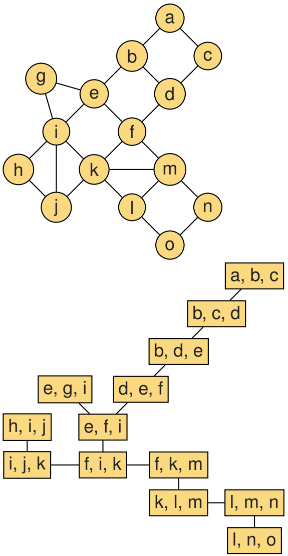

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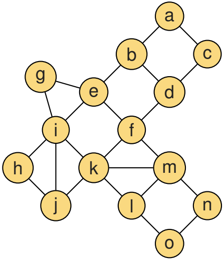
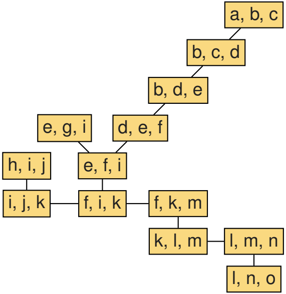
[Robertson & Seymour '84, Arnborg & Proskurowski '89, Bertele & Brioschi '72, Halin '76]





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Theorem (This paper)

There is a $2^{\mathcal{O}(k^2)} n^4$ time algorithm for treewidth.

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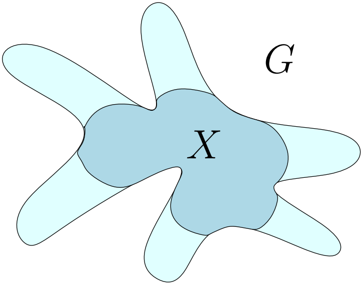

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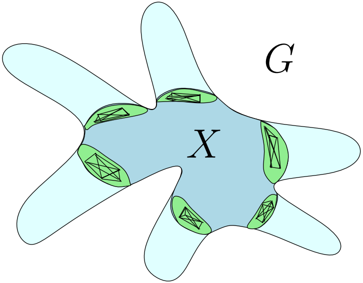

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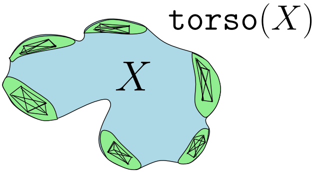

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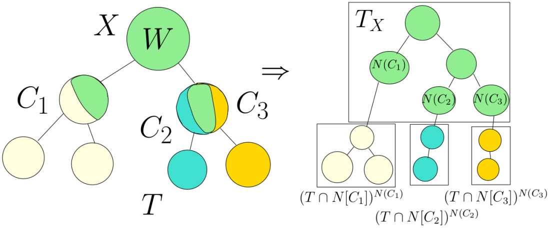

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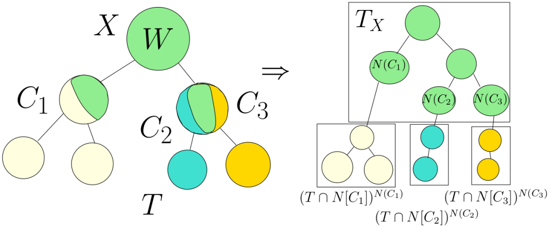
- Decreases the number of largest bags of T



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Open questions:

- Prove a $2^{\Omega(k)}$ lower bound assuming ETH
  - Known reductions give $2^{\Omega(\sqrt{k})}$ lower bound
- Treewidth 1.9-approximation in $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ time?

Thank you!