An Improved Parameterized Algorithm for Treewidth

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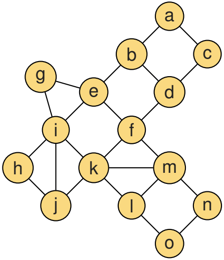
STOC 2023

Treewidth

- Measures how close a graph is to a tree



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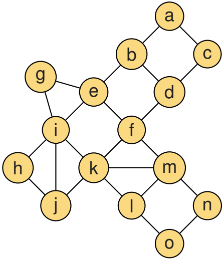
  - Trees have treewidth 1



Treewidth

- Measures how close a graph is to a tree
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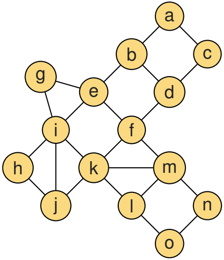
  - The example graph has treewidth 2



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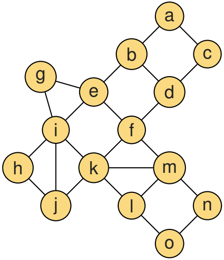

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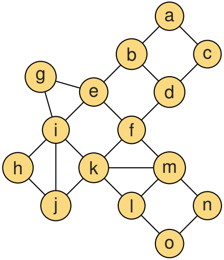

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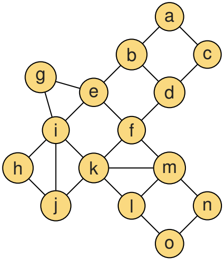

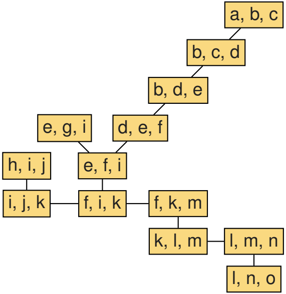

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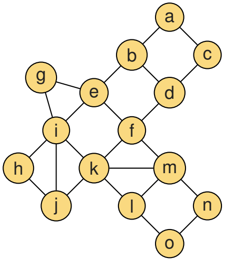

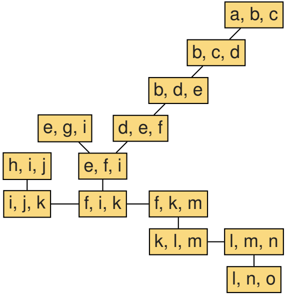

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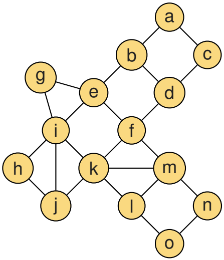

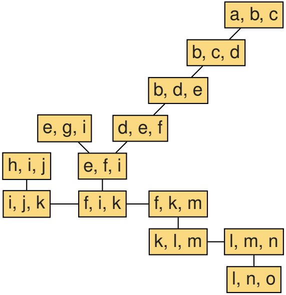

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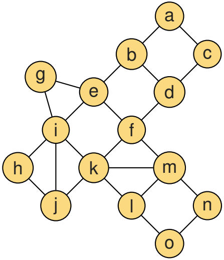

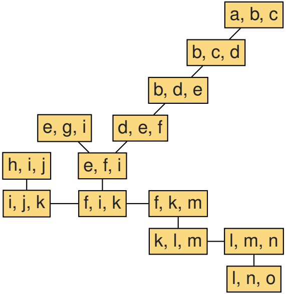

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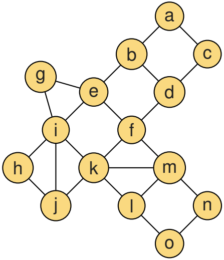

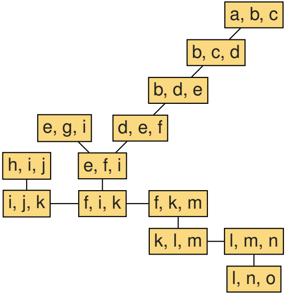

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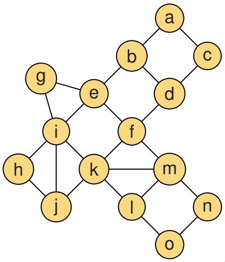

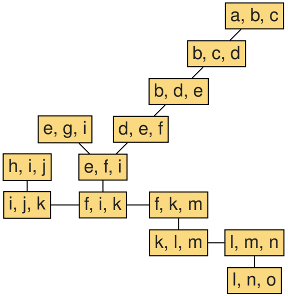

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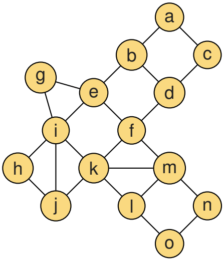
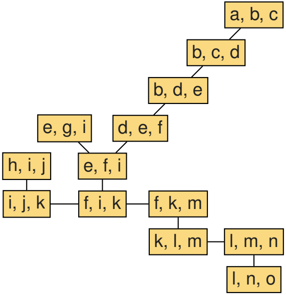
[Robertson & Seymour '84, Arnborg & Proskurowski '89, Bertele & Brioschi '72, Halin '76]





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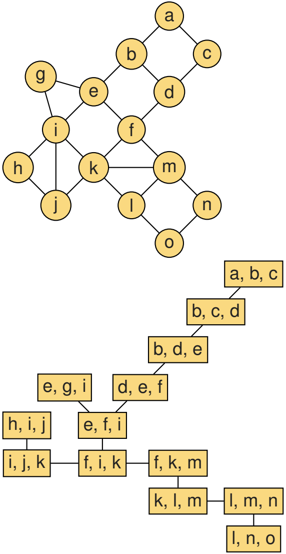

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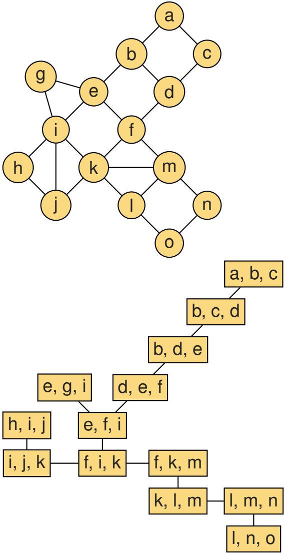

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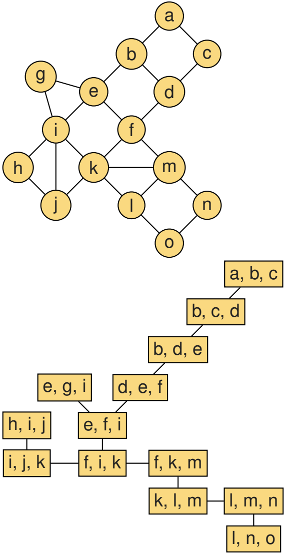

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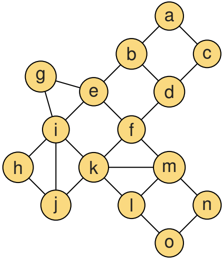

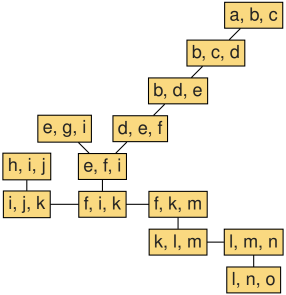

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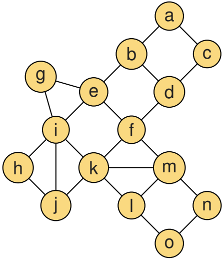

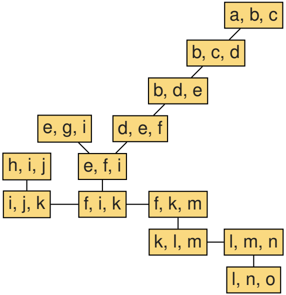

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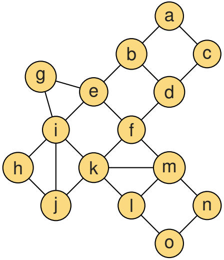

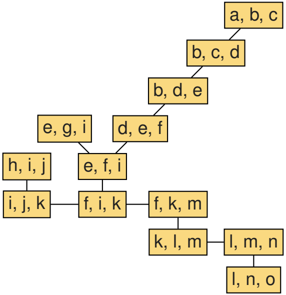

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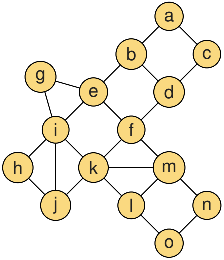

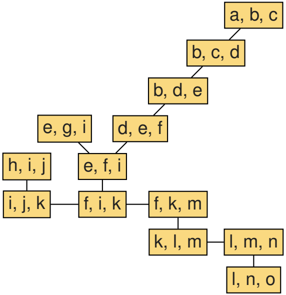

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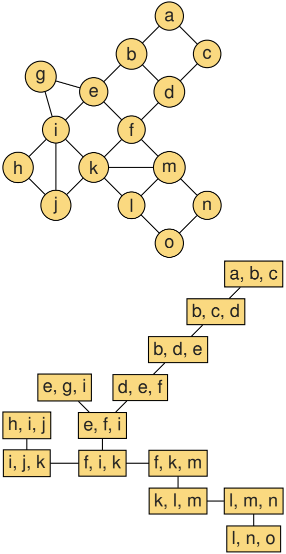

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There is a $2^{\mathcal{O}(k^2)} n^4$ time algorithm for treewidth.

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  - $2^{\mathcal{O}(k)} n$ time 2-approximation [K. '21]

Theorem (This work)

There is a $k^{\mathcal{O}(k/\varepsilon)} n^4$ time $(1+\varepsilon)$-approximation algorithm for treewidth.

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Theorem: It suffices to solve the *Subset treewidth* problem

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Theorem: $2^{\mathcal{O}(k^2)} n^2$ and $k^{\mathcal{O}(k/\varepsilon)} n^2$ time algorithms for Subset treewidth

Techniques:

- Branching on important separators [Marx '06]

Overview of our algorithms

Strategy: Iteratively improve a tree decomposition

1. How to improve a tree decomposition

Theorem: It suffices to solve the *Subset treewidth* problem

Techniques:

- Generalization of the improvement method from [K. '21]
  - Pulling argument to re-arrange tree decompositions, originating from lean tree decompositions [Thomas '90, Bellenbaum and Diestel '02]

2. Solving the subset treewidth problem

Theorem: $2^{\mathcal{O}(k^2)} n^2$ and $k^{\mathcal{O}(k/\varepsilon)} n^2$ time algorithms for Subset treewidth

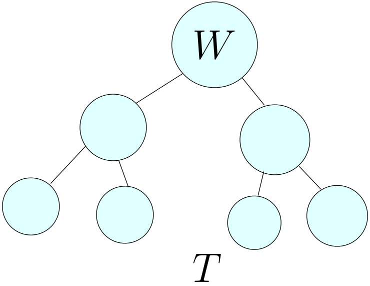
Techniques:

- Branching on important separators [Marx '06]
- Together with the pulling argument

Improving a tree decomposition

Setting

Suppose we have a tree decomposition T whose largest bag is W



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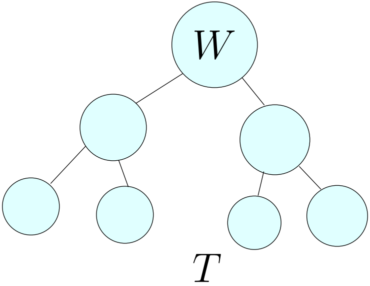

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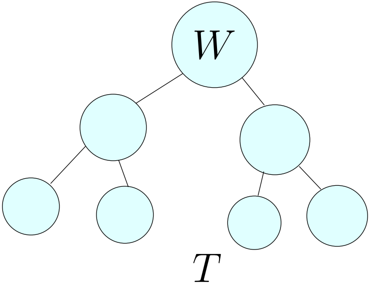
1. either decrease the number of bags of size $|W|$ while not increasing the width of T, or



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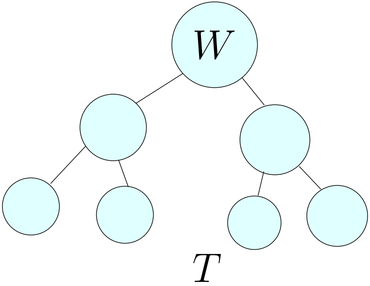
Suppose we have a tree decomposition T whose largest bag is W

Goal:

1. either decrease the number of bags of size $|W|$ while not increasing the width of T, or
2. conclude that T is (approximately) optimal



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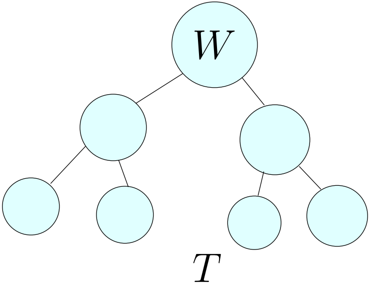
Repeat for $\mathcal{O}(\text{tw}(G) \cdot n)$ iterations to get an (approximately) optimal tree decomposition



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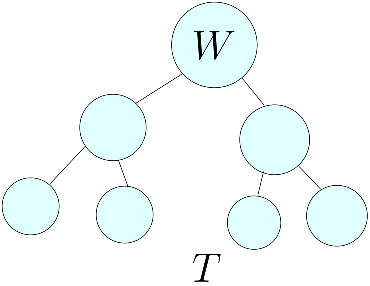
(assume to start with width $\mathcal{O}(\text{tw}(G))$ decomposition)



Improving a tree decomposition

Let W be a largest bag of T

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Want to find:

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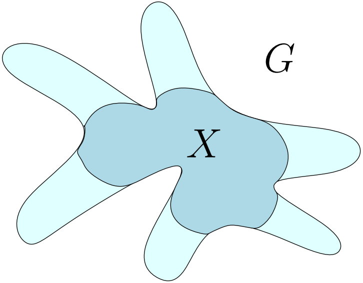
Torso?



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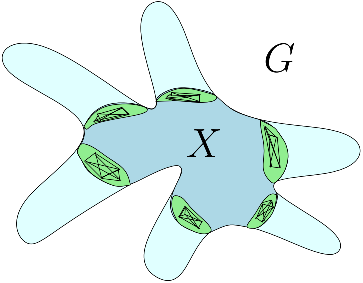

- Make neighborhoods of components of $G \setminus X$ into cliques

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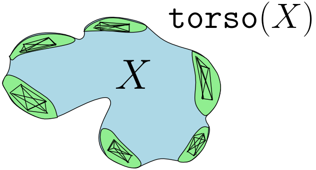

- Make neighborhoods of components of $G \setminus X$ into cliques
- Delete $V(G) \setminus X$

Improving a tree decomposition

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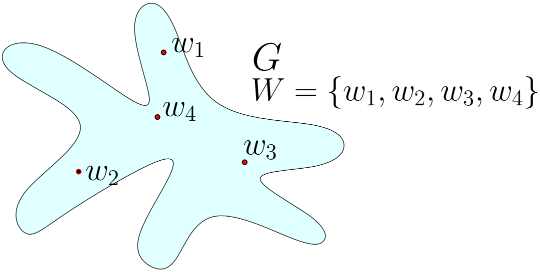

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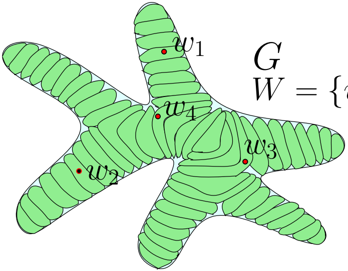

G
$W = \{w_1, w_2, w_3, w_4\}$

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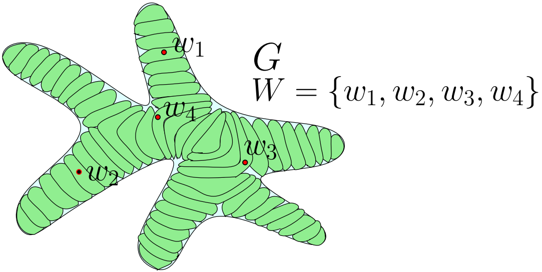

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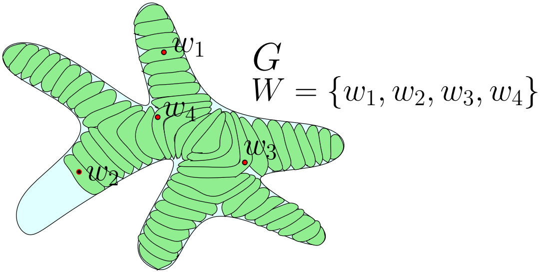

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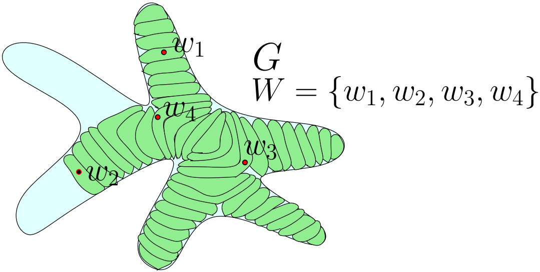

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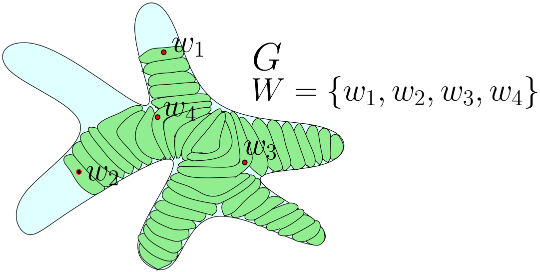

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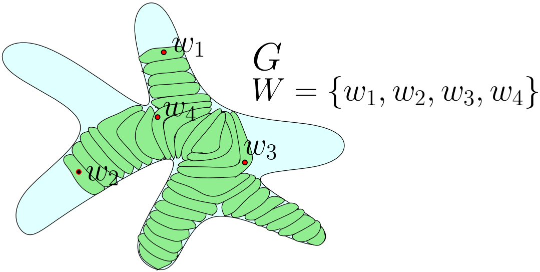

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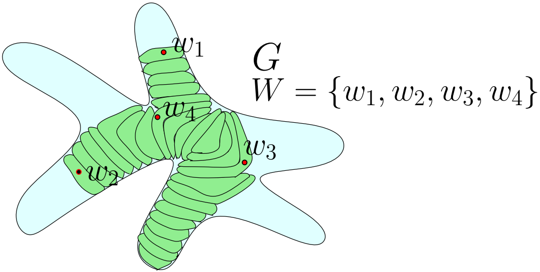

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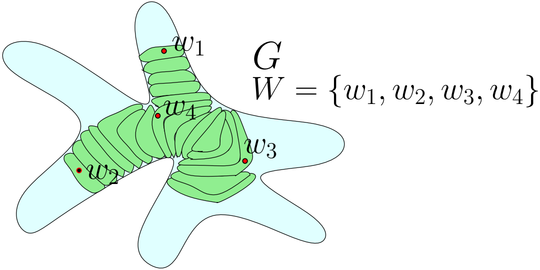

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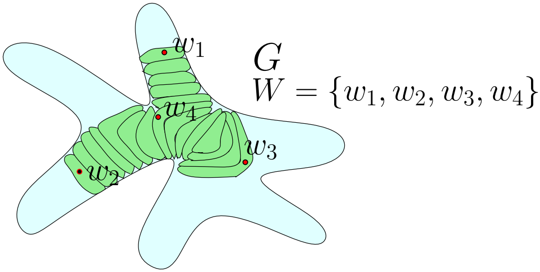

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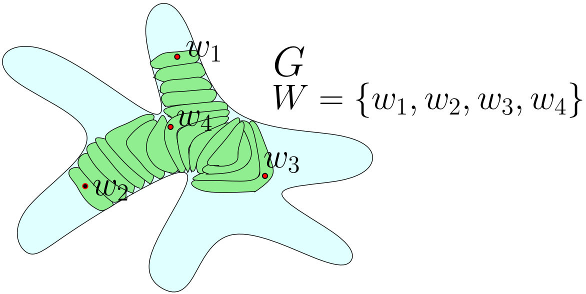
Big-leaf formulation:



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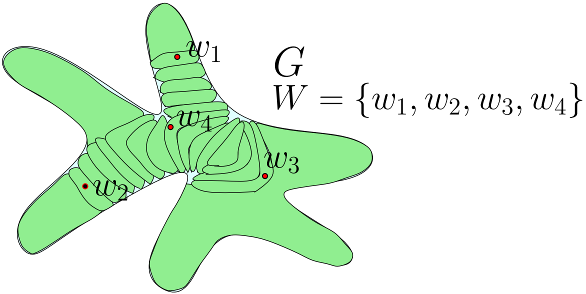
Big-leaf formulation:

- Find a tree decomposition of G whose internal bags have size $\leq |W| - 1$ and cover W, but leaf bags can be arbitrarily large



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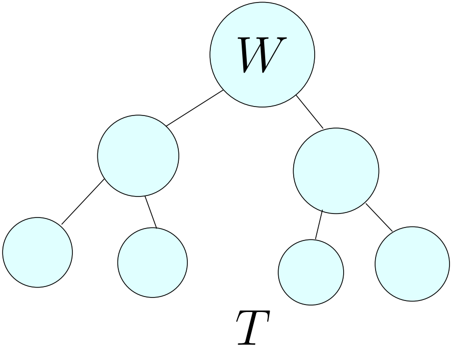
Improving T:



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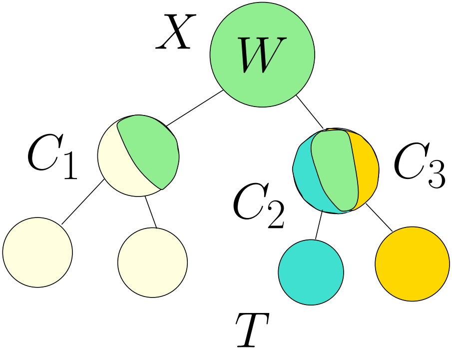

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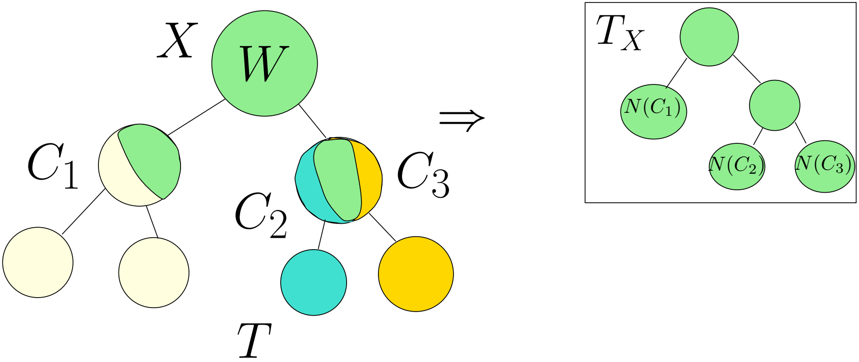

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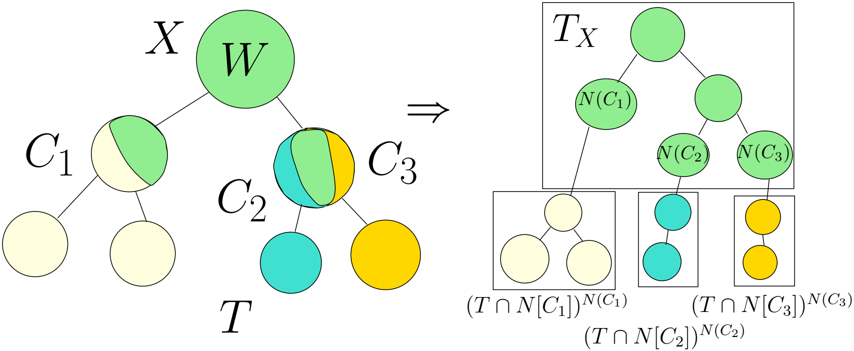

Subset treewidth for exact algorithms

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SUBSET TREEWIDTH

Input: Graph G, integer k, set of vertices $W \subseteq V(G)$ with $|W| = k + 2$

Output: Set $X \subseteq V(G)$ with $W \subseteq X$ and tree decomposition of $\texttt{torso}(X)$ of width $\leq k$ or that the treewidth of G is $> k$

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Theorem

If there is an $f(k) \cdot n^{\mathcal{O}(1)}$ time algorithm for subset treewidth, then there is an $f(k) \cdot n^{\mathcal{O}(1)}$ time algorithm for treewidth with the same function f.

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$2^{\mathcal{O}(k^2)} n^2$ time algorithm for subset treewidth $\rightarrow$ $2^{\mathcal{O}(k^2)} n^4$ time algorithm for treewidth

Subset treewidth for approximation schemes

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PARTITIONED SUBSET TREEWIDTH

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$k^{\mathcal{O}(kt)} n^2$ time algorithm for partitioned subset treewidth $\rightarrow$ $k^{\mathcal{O}(k/\varepsilon)} n^4$ time $(1+\varepsilon)$-approximation algorithm for treewidth

Conclusion

- $2^{\mathcal{O}(k^2)}n^4$ time algorithm and $k^{\mathcal{O}(k/\varepsilon)}n^4$ time $(1+\varepsilon)$-approximation for treewidth

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Open questions:

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  - Known reductions give $2^{\Omega(\sqrt{k})}$ lower bound

Thank you!

- Longer talk: https://www.youtube.com/watch?v=9If417oeWcQ
- Slides: https://tuukkakorhonen.com/