Finding sparse induced subgraphs on graphs of bounded induced matching treewidth

Hans L. Bodlaender[1], Fedor V. Fomin[2], and Tuukka Korhonen

UNIVERSITY OF COPENHAGEN

[1]Utrecht University, [2]University of Bergen

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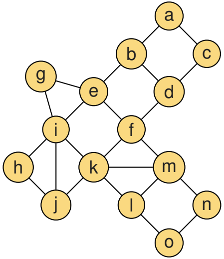
Treewidth



Graph G

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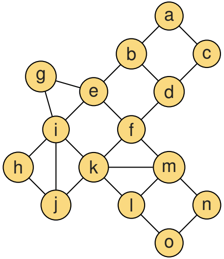

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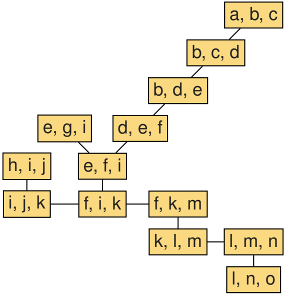

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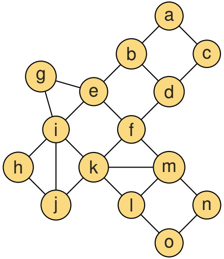

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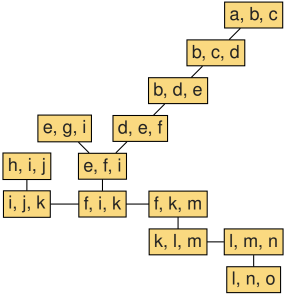

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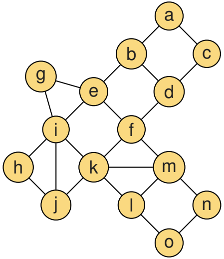

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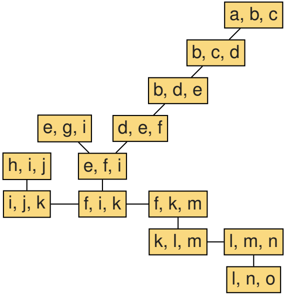

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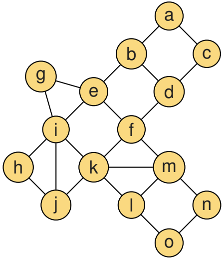

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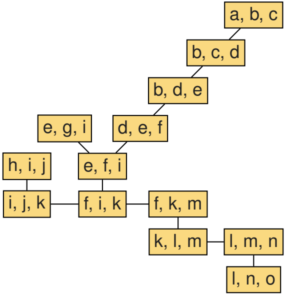

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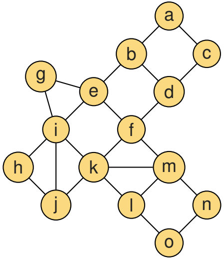

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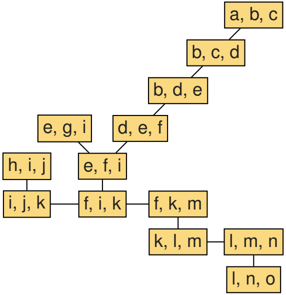

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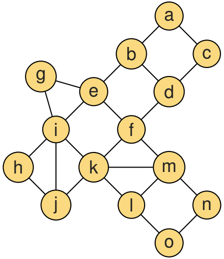

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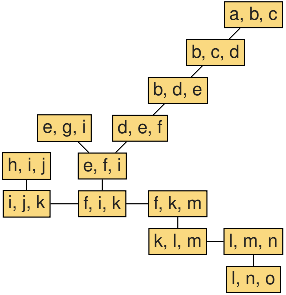

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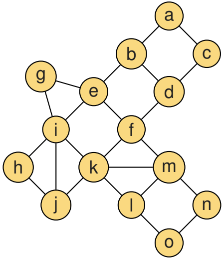

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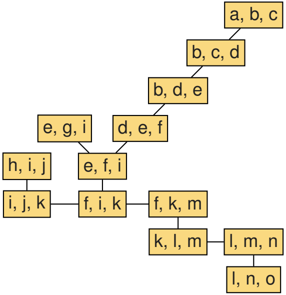

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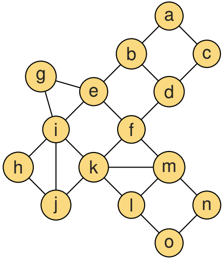

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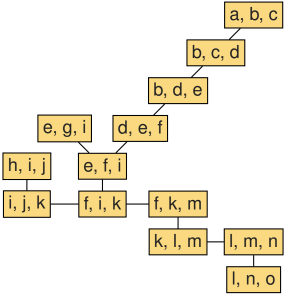

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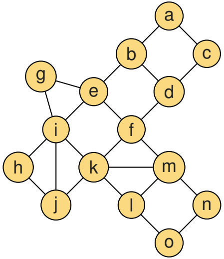

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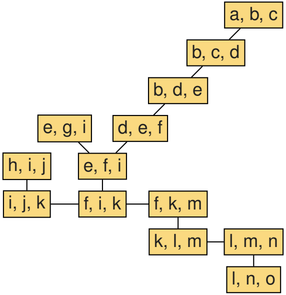

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[Robertson & Seymour'84, Arnborg & Proskurowski'89, Bertele & Brioschi'72, Halin'76]

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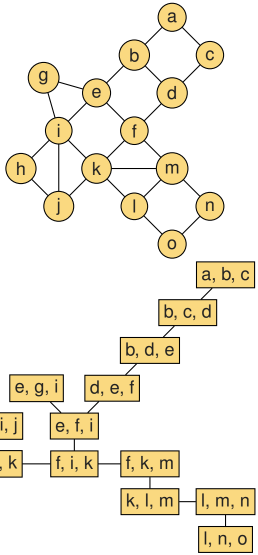

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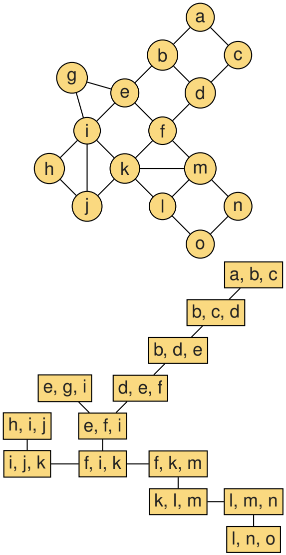

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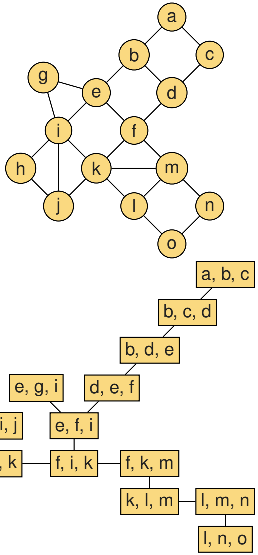

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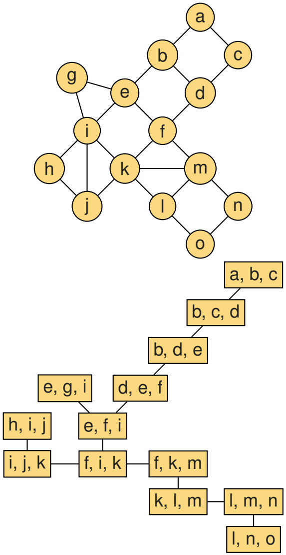

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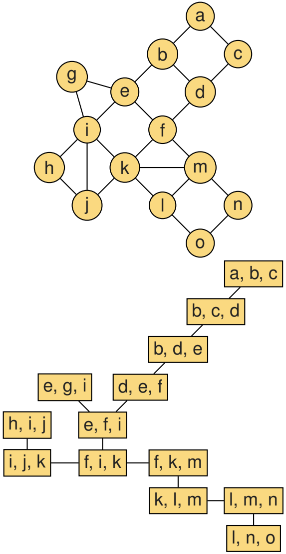

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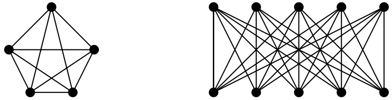

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Given an n-vertex graph G with tree-$\mu(G) \leq k$, a **CMSO$_2$**-sentence Φ, and an integer w, we can in time $f(k, w, |\Phi|) \cdot n^{\mathcal{O}(kw^2)}$ find a maximum-size set $X \subseteq V(G)$ so that

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Implies $f(k) \cdot n^{\mathcal{O}(k)}$ time algorithms for

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Answers an open problem of [Lima, Milanič, Muršič, Okrasa, Rzążewski, and Štorgel, '24], who gave a similar theorem for tree-independence number

The algorithm

Step 1: Containers

Yolov's idea:

Lemma (Yolov'18)

For a set $B \subseteq V(G)$ with $\mu(B) \leq k$, there are at most $n^{\mathcal{O}(k)}$ possible intersections $B \cap I$, where I is a maximal independent set of G.

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- guess the vertex cover
- use Yolov's lemma, a "kicking out"-lemma, and properties of **CMSO$_2$** to guess the rest

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Let T be a nice tree decomposition of G with $\mu(T) \leq k$, and $X \subseteq V(G)$ a set with $\mathrm{tw}(G[X]) \leq w$. There is a tree decomposition T_X of $G[X]$ of width $\mathcal{O}(kw^2)$, whose structure "follows" T.

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- There is a partition (X_1, X_2) of X so that:
  - Vertices in X_1 extend their appearances upwards compared to T
  - Vertices in X_2 form an independent set and each appears only in its "top bag"
- This allows to run a dynamic programming on T, that simultaneously:
  - Guesses the set X
  - Guesses the decomposition T_X
  - Evaluates Φ on $G[X]$ by using T_X

Conclusion

- An $f(k, w, |\Phi|) \cdot n^{\mathcal{O}(kw^2)}$ time algorithm for finding maximum induced subgraph of treewidth $\leq w$ satisfying Φ on graphs of tree-$\mu(G) \leq k$

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Thank you!