An Improved Parameterized Algorithm for Treewidth

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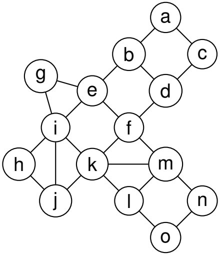
UNIVERSITY OF BERGEN

based on joint work with Daniel Lokshtanov, UCSB

Princeton Discrete Mathematics Seminar

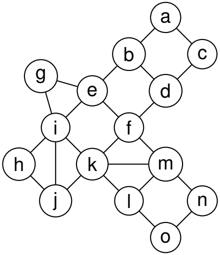
2 March 2023

Treewidth



Graph G

Treewidth



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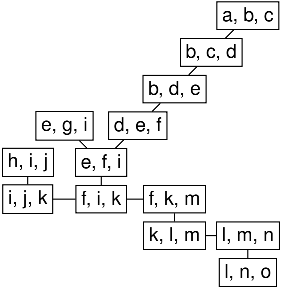
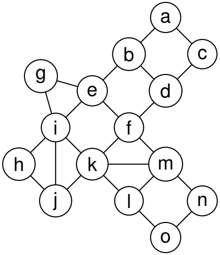


A tree decomposition of G

Treewidth



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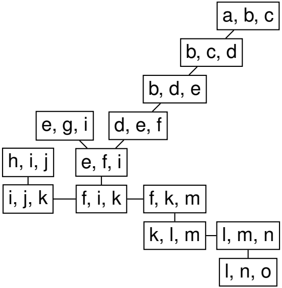

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1. Every vertex should be in a bag

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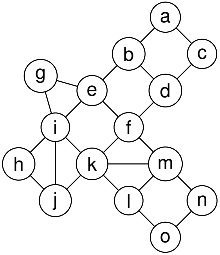

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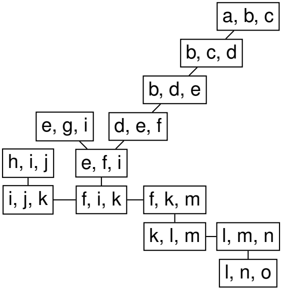

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1. Every vertex should be in a bag
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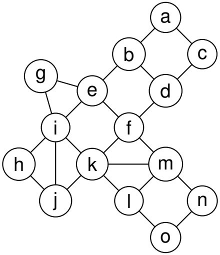

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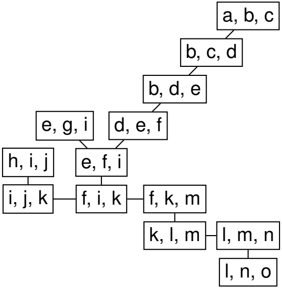

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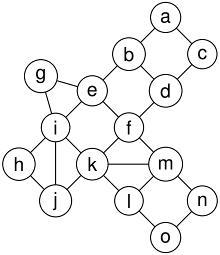

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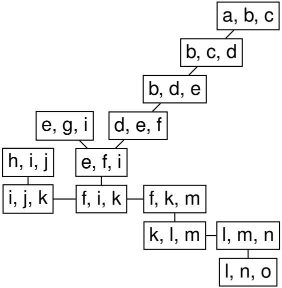

A tree decomposition of G

1. Every vertex should be in a bag
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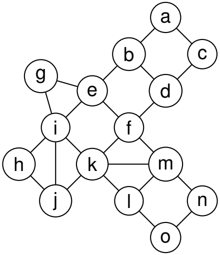

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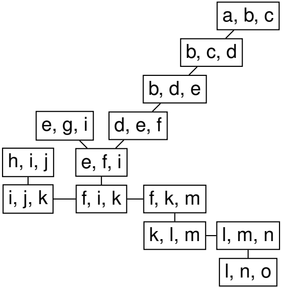

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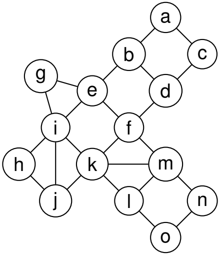

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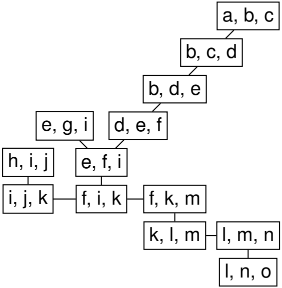

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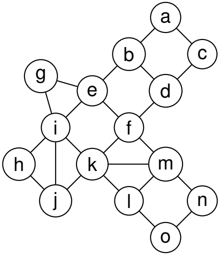

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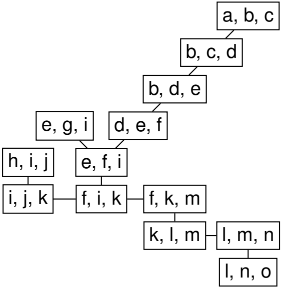

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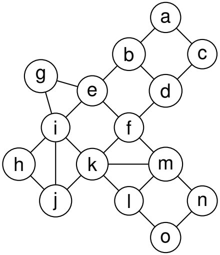

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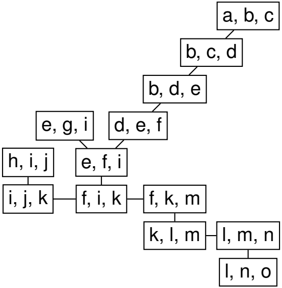

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[Robertson & Seymour'84, Bertele & Brioschi'72, Halin'76]

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There is a $2^{\mathcal{O}(k^2)} n^4$ time algorithm for treewidth.

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Theorem (K. & Lokshtanov '23)

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- No dynamic programming, runs in space poly(n, k)

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Theorem (K. & Lokshtanov '23)

There is a $k^{\mathcal{O}(k/\varepsilon)} n^4$ time $(1+\varepsilon)$-approximation algorithm for treewidth.

Outline

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1. How to improve a tree decomposition

Suffices to solve the *Subset treewidth* problem

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Suffices to solve the *Subset treewidth* problem

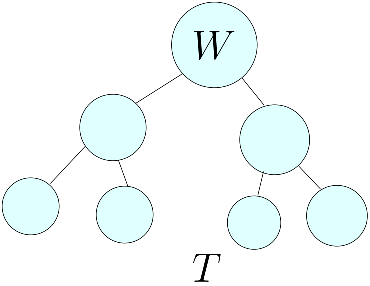
2. Solving the subset treewidth problem

Algorithms for subset treewidth that then imply algorithms for treewidth

How to improve a tree decomposition

Setting

Suppose we have a tree decomposition T whose largest bag is W



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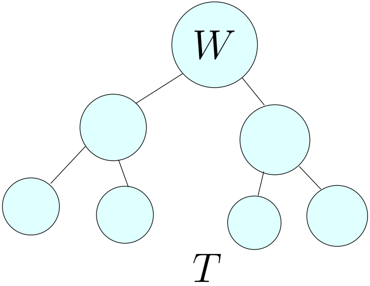

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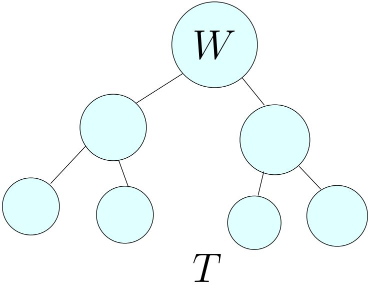

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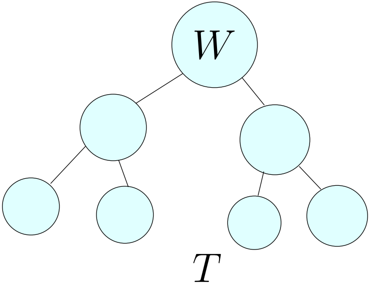

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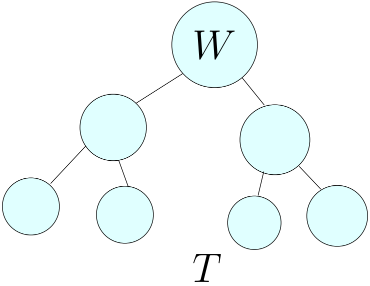

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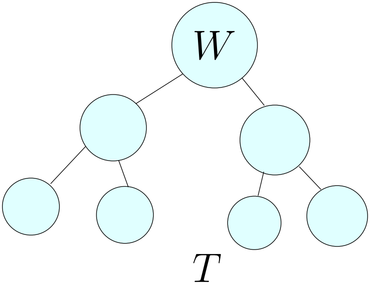
(assume to start with width $\mathcal{O}(\text{tw}(G))$ decomposition)



Improving a tree decomposition

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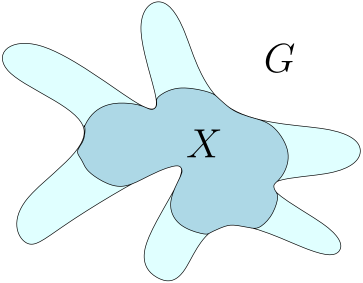

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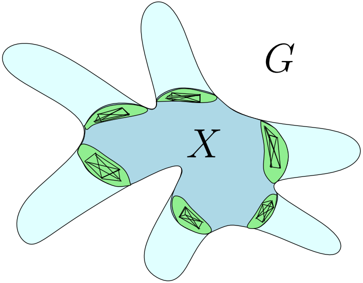

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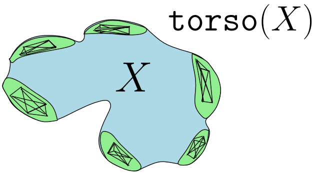

- Make neighborhoods of components of $G \setminus X$ into cliques
- Delete $V(G) \setminus X$

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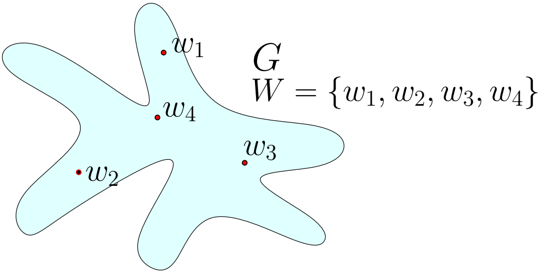

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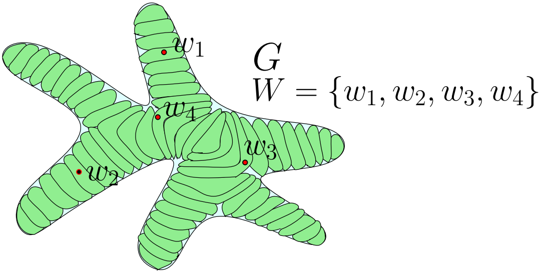

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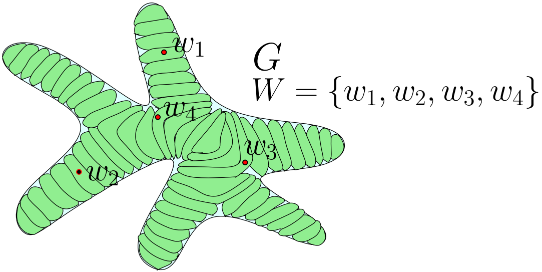

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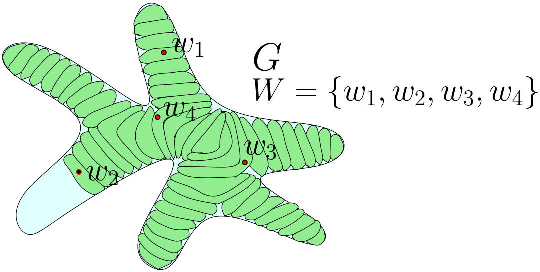

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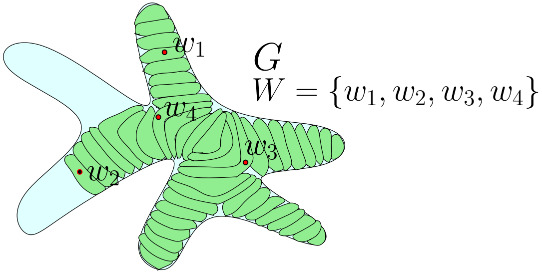

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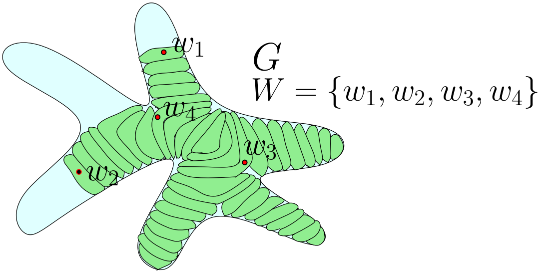

Improving a tree decomposition

Let W be a largest bag of T — SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
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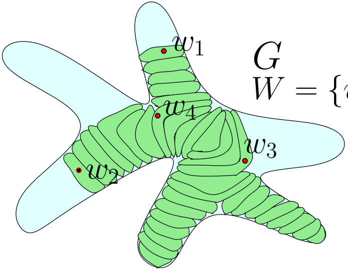

G

$W = \{w_1, w_2, w_3, w_4\}$

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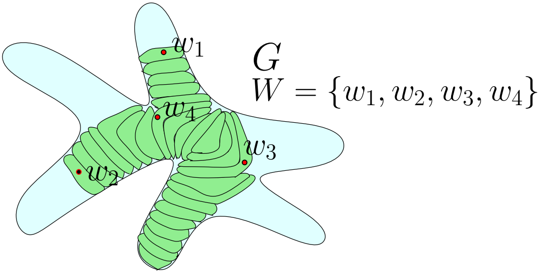

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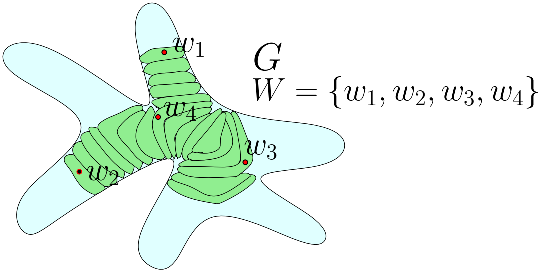

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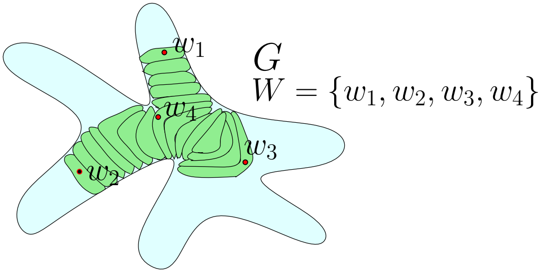

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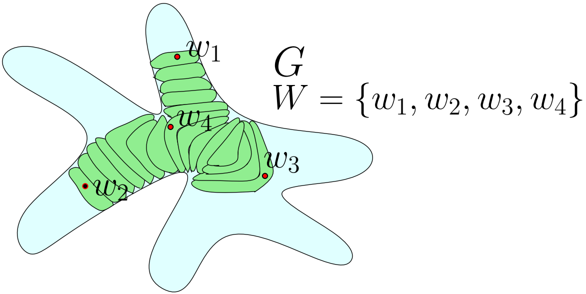

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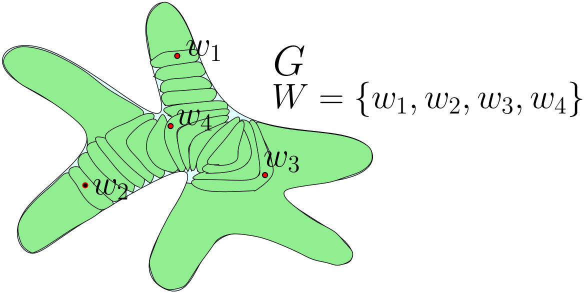
Big-leaf formulation:

- Find a tree decomposition of G whose internal bags have size $\leq |W| - 1$ and cover W, but leaf bags can be arbitrarily large



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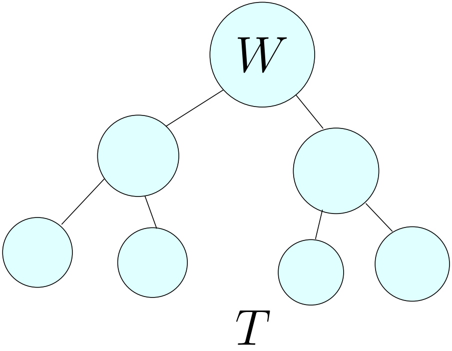

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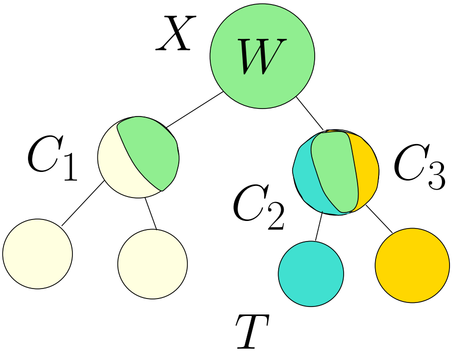

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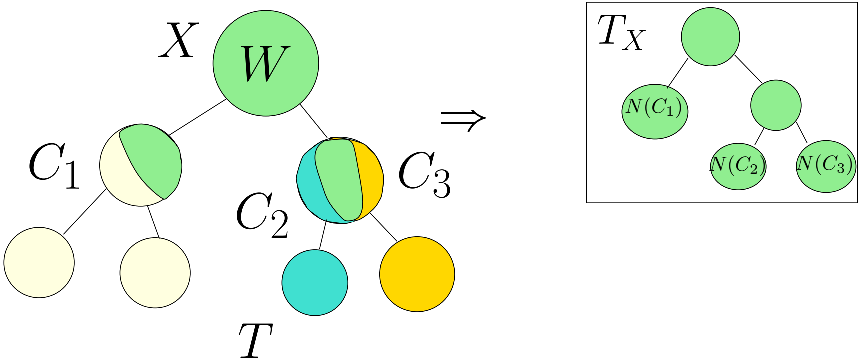

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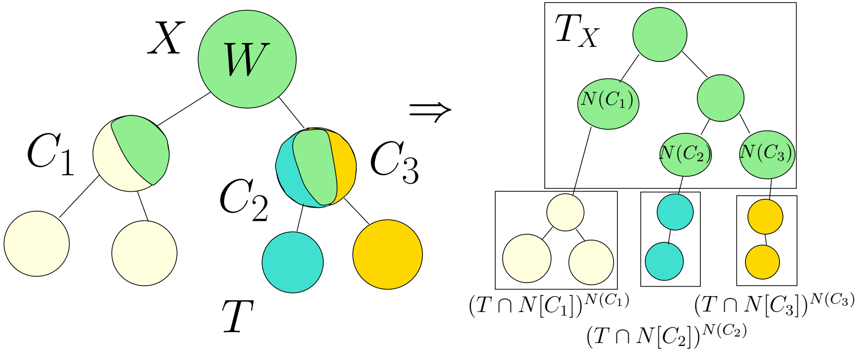

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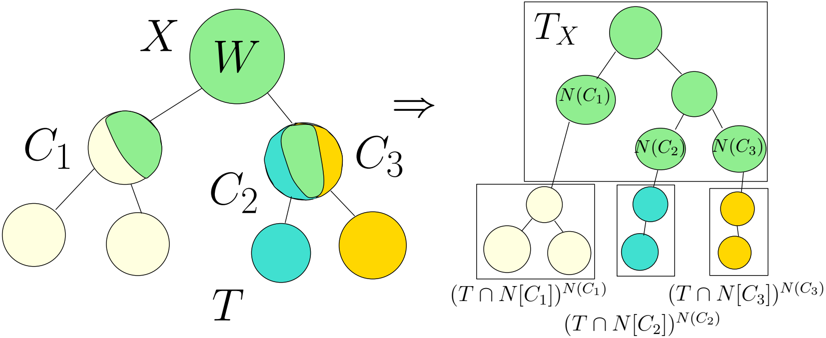

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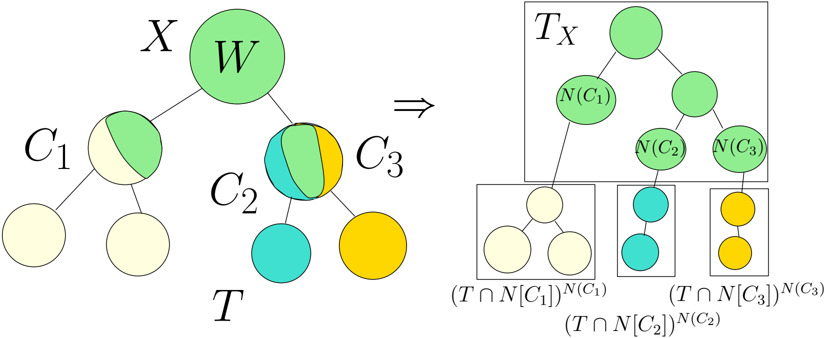

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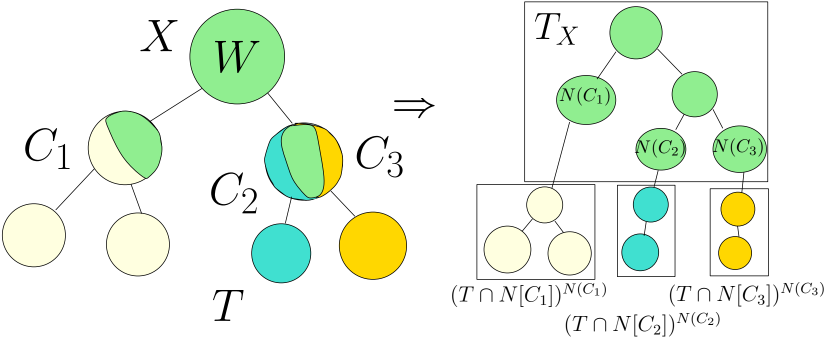

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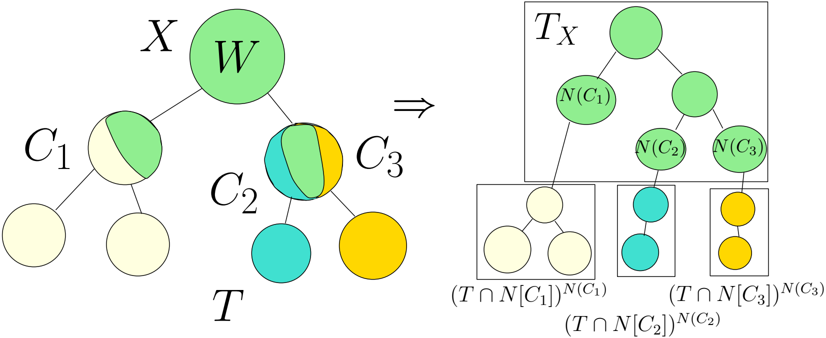

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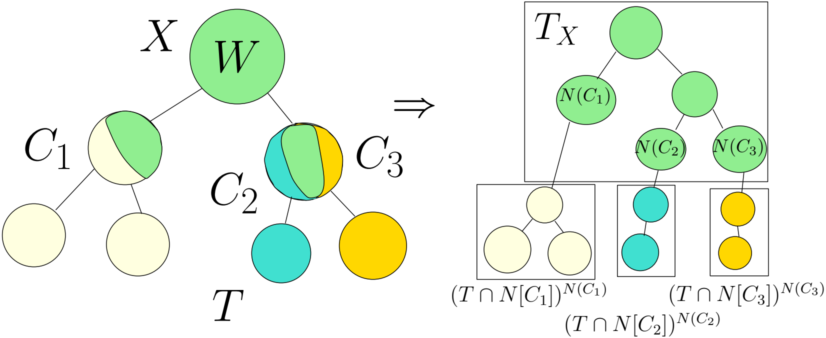

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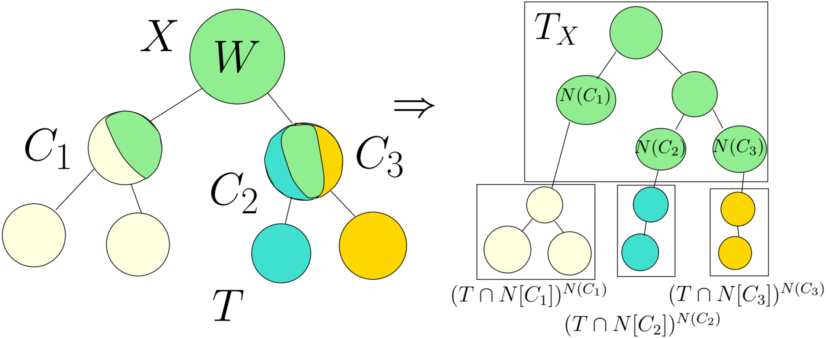

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- (actually need a bit stronger condition than linkedness for improvement)

Subset treewidth for exact algorithms

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SUBSET TREEWIDTH

Input: Graph G, integer k, set of vertices $W \subseteq V(G)$ with $|W| = k + 2$

Output: Set $X \subseteq V(G)$ with $W \subseteq X$ and tree decomposition of $\texttt{torso}(X)$ of width $\leq k$ or that the treewidth of G is $> k$

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If there is an $f(k) \cdot n^{\mathcal{O}(1)}$ time algorithm for subset treewidth, then there is an $f(k) \cdot n^{\mathcal{O}(1)}$ time algorithm for treewidth with the same function f.

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$2^{\mathcal{O}(k^2)} n^2$ time algorithm for subset treewidth $\rightarrow 2^{\mathcal{O}(k^2)} n^4$ time algorithm for treewidth

Subset treewidth for approximation schemes

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(this is also a $k^{\mathcal{O}(k^2)} n^{\mathcal{O}(1)}$ time algorithm for subset treewidth)

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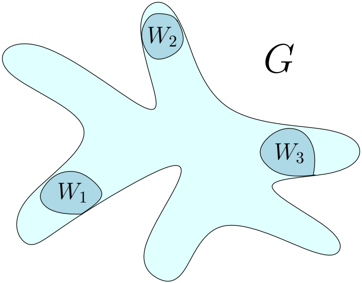

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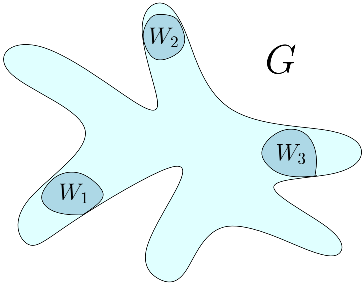

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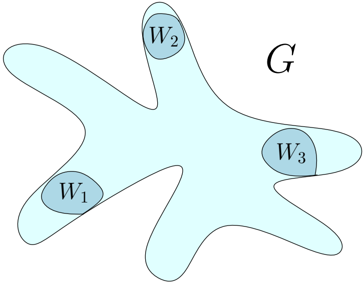

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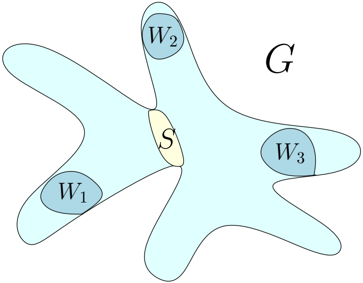

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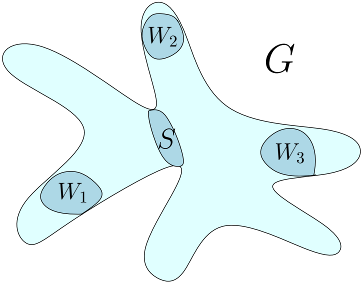

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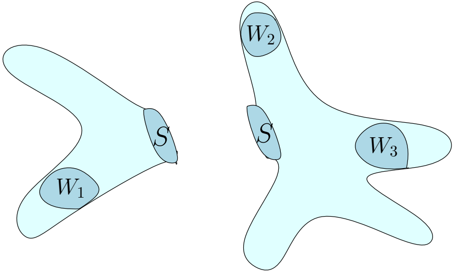
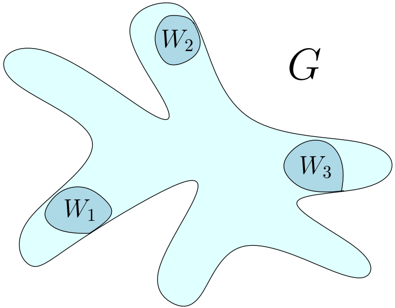


Branching for partitioned subset treewidth

- Now terminal cliques *strongly linked* into each other



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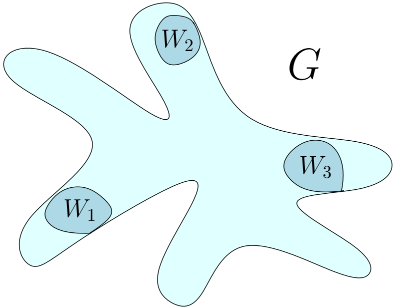

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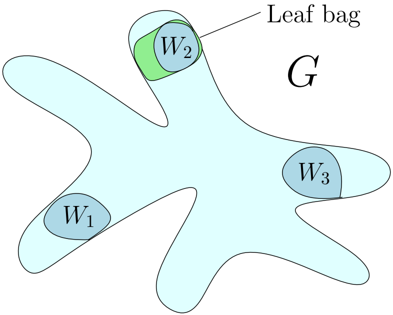

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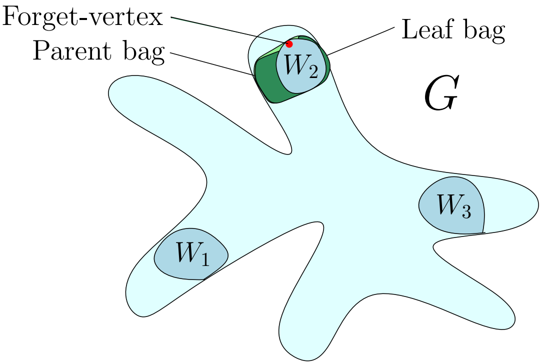

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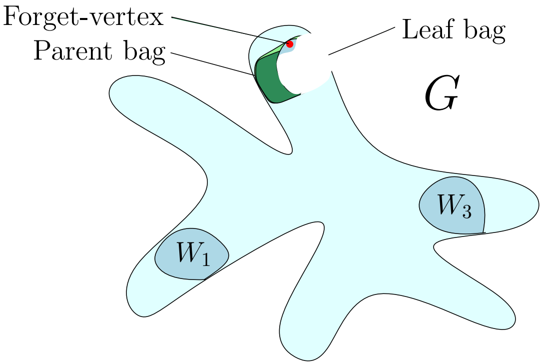

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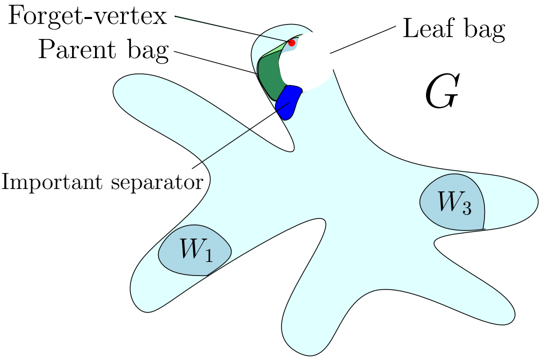

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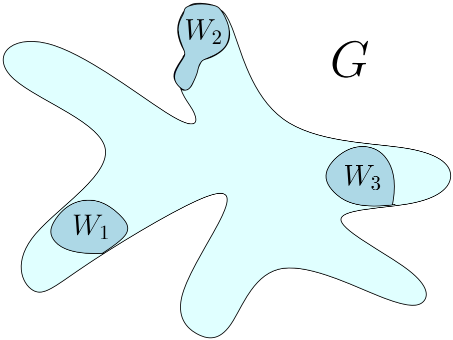
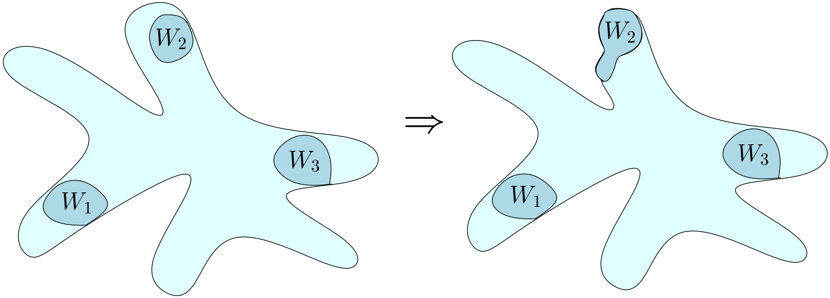


Analysis of branching



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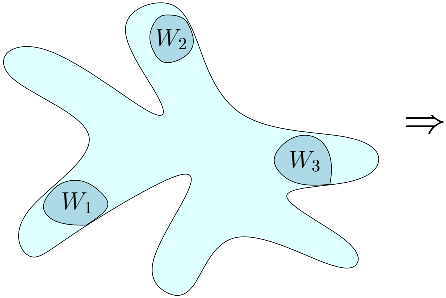

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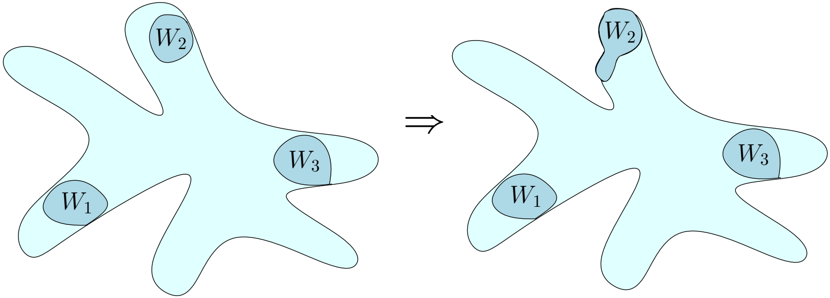

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- To get $k^{\mathcal{O}(kt)} n^{\mathcal{O}(1)}$ time, need also an important separator hitting set lemma

Conclusion

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