Tutorial: Dynamic FPT Algorithms

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Workshop on Modern Trends in Parameterized Complexity

Outline

- ⇒ Intro
- Technical showcase: Dynamic Baker's scheme
- Overview of the literature
- Technical showcase: Dynamic treewidth
- Future of dynamic FPT and a mini open problem / brainstorming session

Dynamic graph algorithms

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- Observation: Dynamic XP not interesting
  - Sublinear time for every fixed $k \Rightarrow$ FPT algorithm

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- [Iwata & Oka '14]: $\mathcal{O}(k^2)$ update time $\mathcal{O}(k^2)$ kernel
- [Alman, Mnich & Vassilevska Williams '17]: $\mathcal{O}(1)$ amortized update time $\mathcal{O}(k^2)$ kernel

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Theorem (Baker '83)

There is a $(1-\varepsilon)$-approximation for maximum weight independent set (MWIS) on planar graphs, with running time $2^{\mathcal{O}(1/\varepsilon)}n$.

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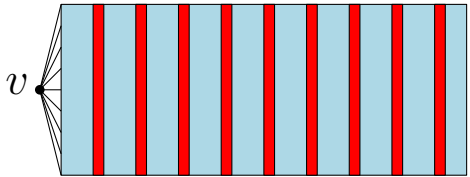

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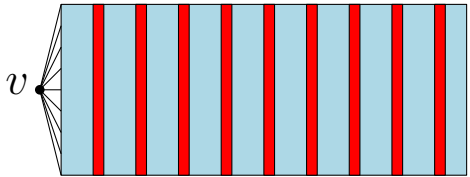

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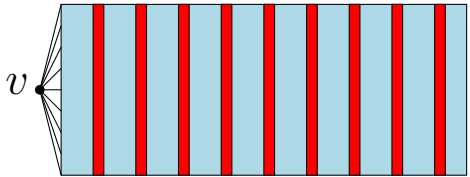

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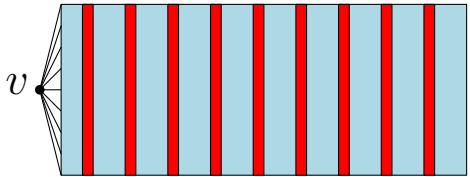

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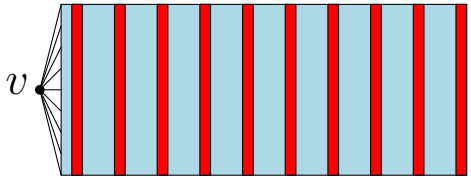

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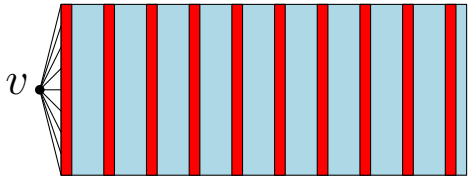

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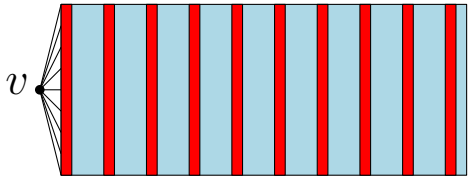

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- Try all offsets $\rightarrow$ For some offset, we delete $\leq 1/\varepsilon$ fraction of optimal solution
- Each can be solved exactly in time $2^{\mathcal{O}(k)}n = 2^{\mathcal{O}(1/\varepsilon)}n$ using treewidth

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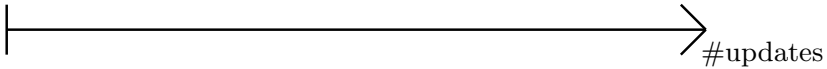

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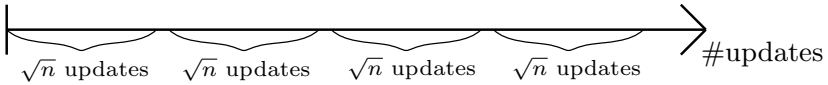

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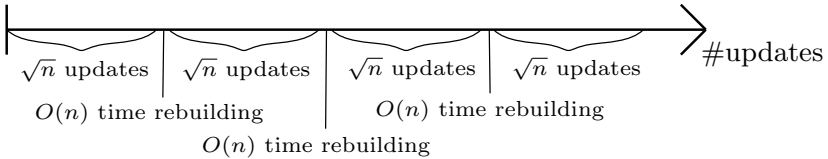

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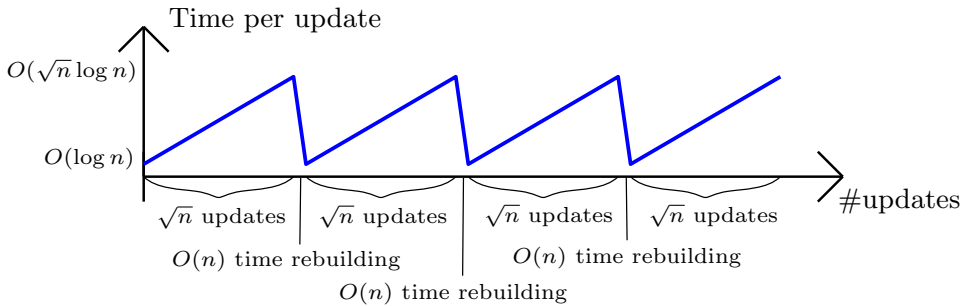

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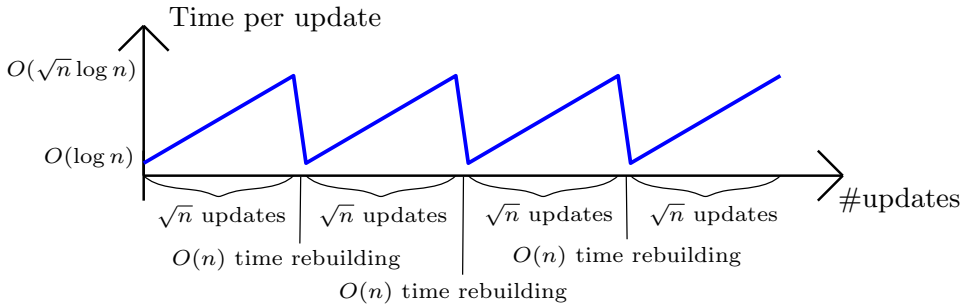

Suffices to design a data structure that:

- Can be initialized in $f(\varepsilon)n$ time
- Has update time $f(\varepsilon) \cdot u \log n$, where u is the number of updates so far

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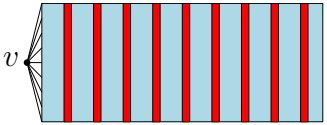

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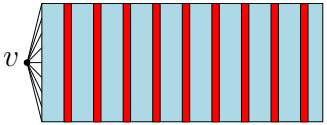

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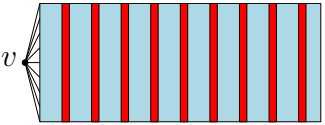
- After arbitrary edge updates $\exists i$ s.t. $\mathsf{OPT}(G[V_i]) \geq (1-\varepsilon) \cdot \mathsf{OPT}(G)$



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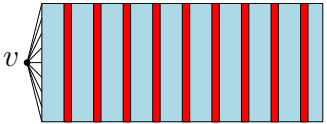
$\Rightarrow$ After initialization, for each V_i a separate data structure that handles the induced subgraph $G[V_i]$



Dynamic Baker: Initialization

Initialization: BFS-layering to compute $k = \lceil 1/\varepsilon \rceil$ vertex sets $V_1, V_2, \ldots, V_k \subseteq V(G)$ so that

- Initially $\mathrm{tw}(G[V_i]) \leq \mathcal{O}(k)$
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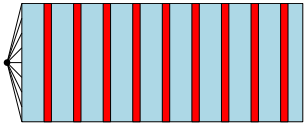

Suffices to design data structure that:

- Is initialized with a weighted planar graph of treewidth $\mathcal{O}(k)$ and a parameter ε in $f(k,\varepsilon) \cdot n$ time
- Maintains $(1-\varepsilon)$-approximation of MWIS (assuming the graph remains planar)
- Has update time $f(k,\varepsilon) \cdot u \log n$, where u the number of updates so far

Dynamic Baker: The main idea

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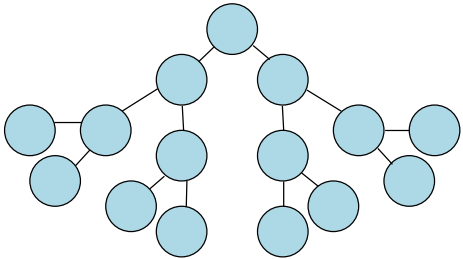

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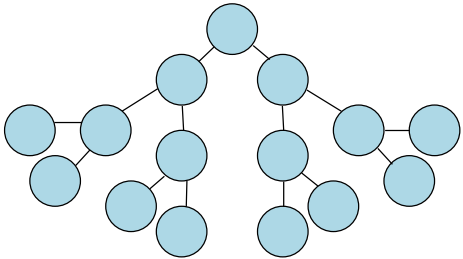

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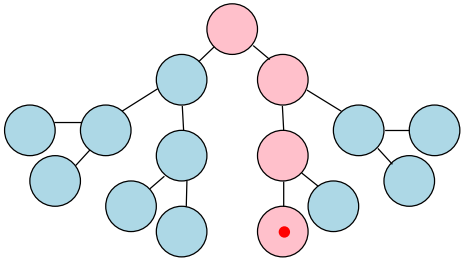

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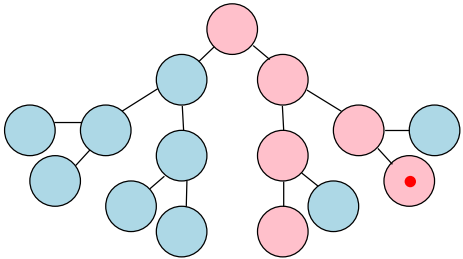

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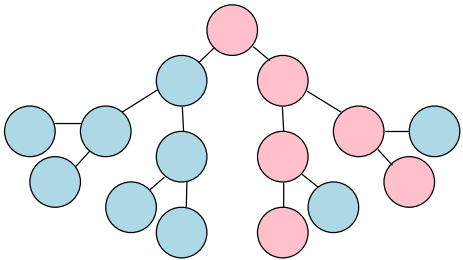

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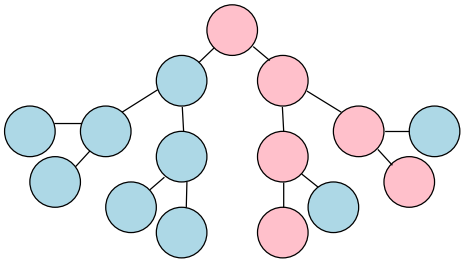

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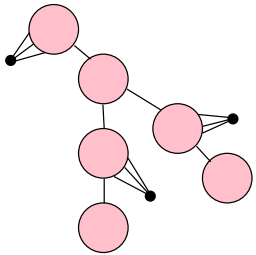

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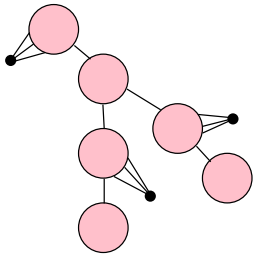

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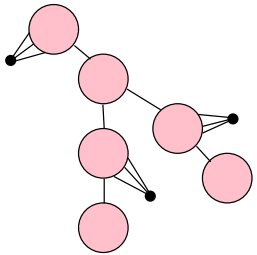

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- $\approx \log \log n$ levels of recursion $\Rightarrow f(\varepsilon) \cdot n^{o(1)}$ update time [K., Nadara, Pilipczuk & Sokołowski '24]

Dynamic Baker: De-amortization

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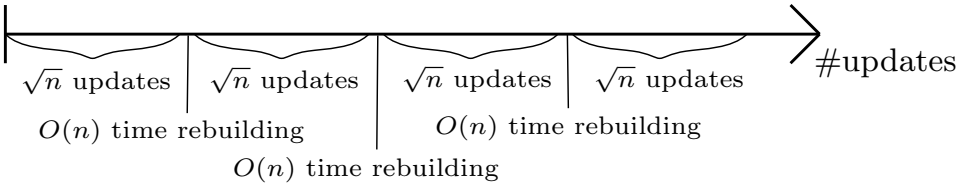

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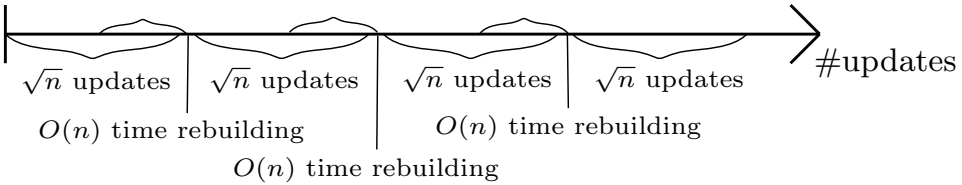

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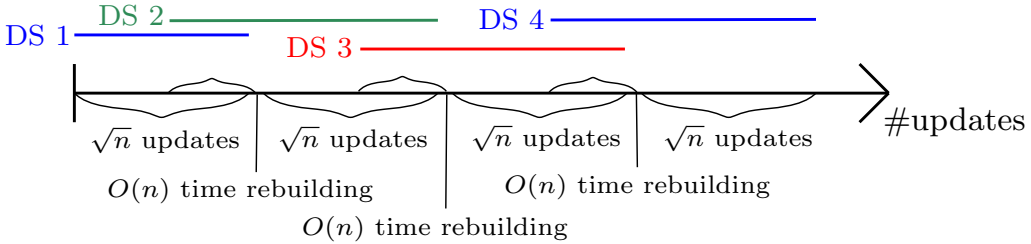

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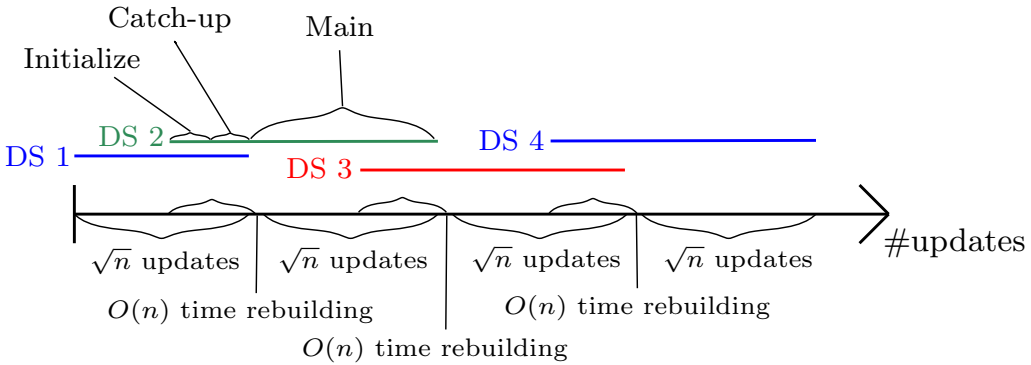

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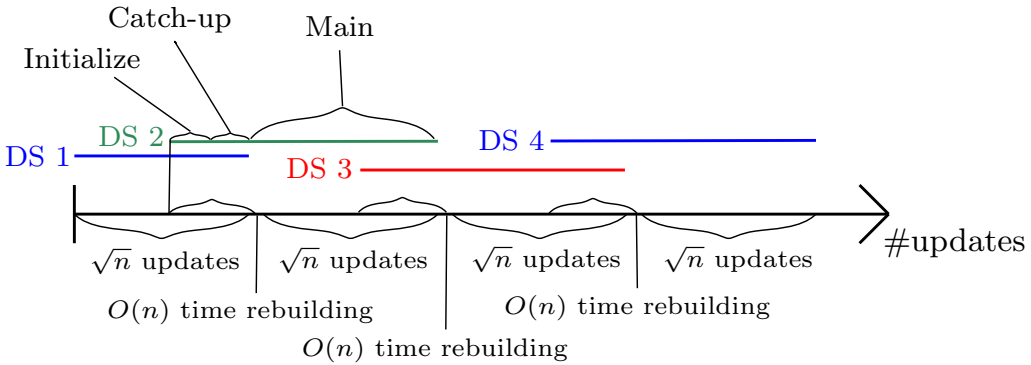

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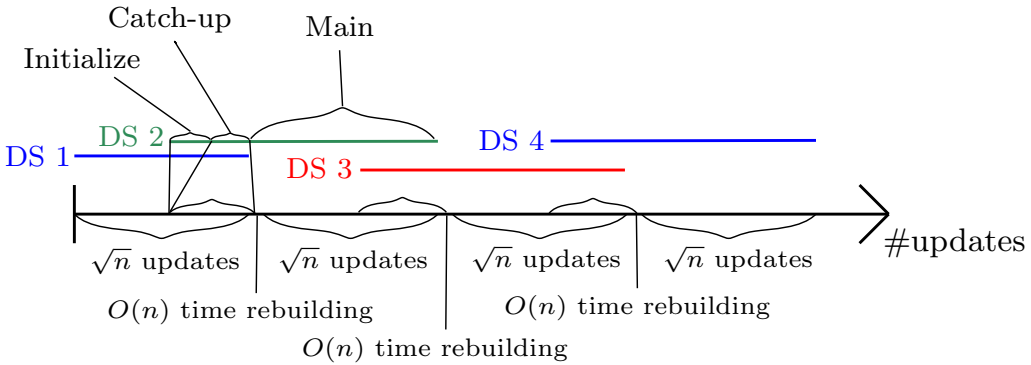

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There is a dynamic $(1-\varepsilon)$-approximation algorithm for MWIS on planar graphs, supporting edge insertions and deletions, and vertex weight-updates, with worst-case update time $f(\varepsilon) \cdot n^{o(1)}$.

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Outline

- Intro
- Technical showcase: Dynamic Baker's scheme

⇒ Overview of the literature

- Technical showcase: Dynamic treewidth
- Future of dynamic FPT and a mini open problem / brainstorming session

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- [Iwata, Oka '14], [Alman, Mnich, Vassilevska Williams '17]: Vertex cover, FVS, k-path, etc.
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- [Bodlaender '93]: Treewidth-2 and Courcelle in that setting
- [Dvorak, Kupec & Tuma '14], [Chen, Czerwinski, Disser, Feldmann, Hermelin, Nadara, Pilipczuk, Pilipczuk, Sorge, Wroblewski & Zych-Pawlewicz '21]: Treedepth, with Courcelle
- [Majewski, Pilipczuk & Sokolowski '23]: FVS with Courcelle
- [Dvorak & Tuma '13]: Counting induced subgraphs on bounded-expansion and nowhere dense graphs

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- [Bertram, Haun, Jensen, K. '26]: Dynamic version of the meta-kernelization framework of [Bodlaender, Fomin, Lokshtanov, Penninkx, Saurabh & Thilikos '09]

Dynamic treewidth: Applications

- [K. & Sokołowski '24]: Dynamic rankwidth with $f(k) \cdot n^{o(1)}$ amortized update time
  - Main application: Static rankwidth in $f(k) \cdot n^{1+o(1)} + \mathcal{O}(m)$
- [K., Pilipczuk & Stamoulis '24]: static k-disjoint paths and H-minor-testing in $f(k) \cdot n^{1+o(1)}$ time
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  - Linear kernels on minor-free graphs, with amortized update time $\mathcal{O}(\log n)$, and worst-case $\mathcal{O}(1)$ updates to the kernel per update to the graph

Outline

- Intro
- Technical showcase: Dynamic Baker's scheme
- Overview of the literature
- ⇒ Technical showcase: Dynamic treewidth
- Future of dynamic FPT and a mini open problem / brainstorming session

Dynamic treewidth: Theorem statement

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There is data structure that

- is initialized with integer k and an edgeless n-vertex graph G
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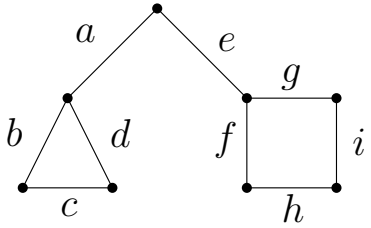
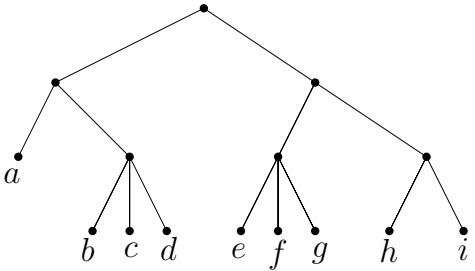
$\Rightarrow$ Dynamic Courcelle's theorem in $f(k) \cdot \log n$ amortized update time

Dynamic treewidth: Maintained decomposition





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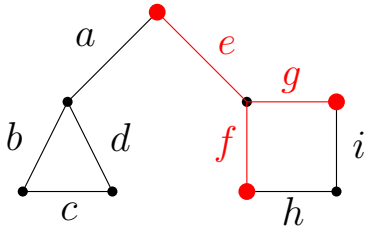

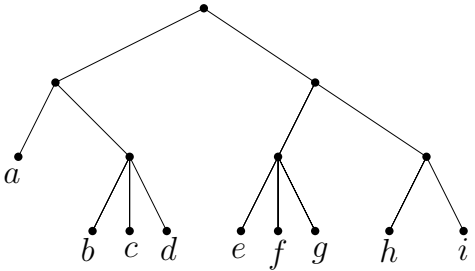

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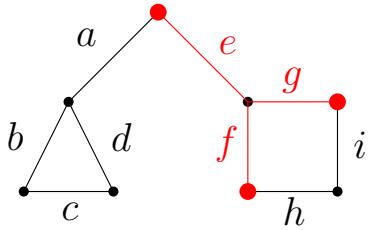

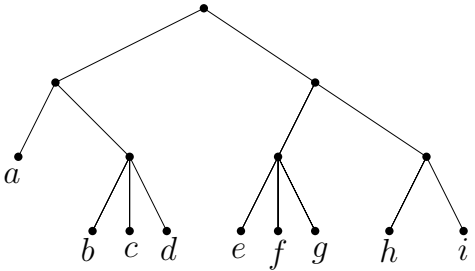

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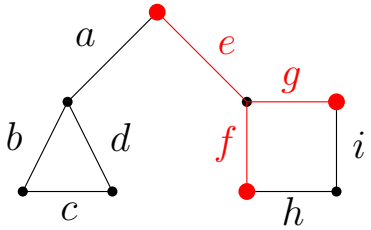

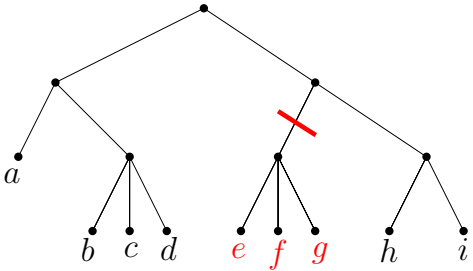

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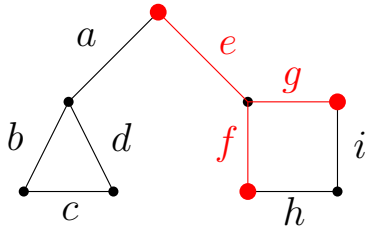

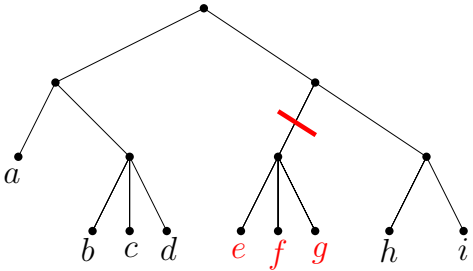

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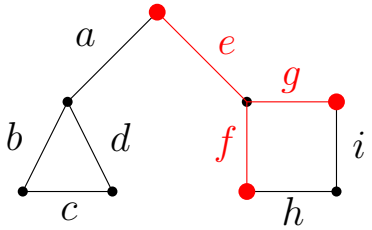

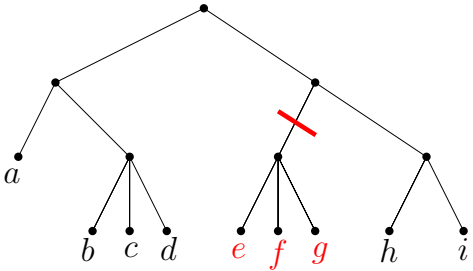

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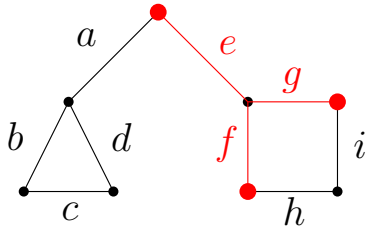

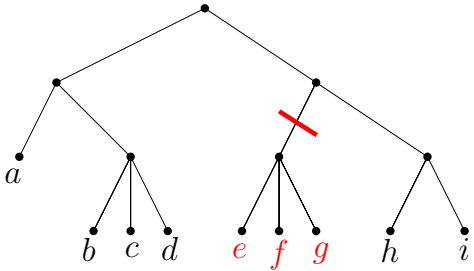

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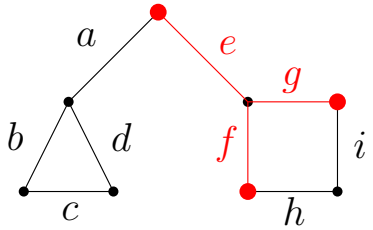

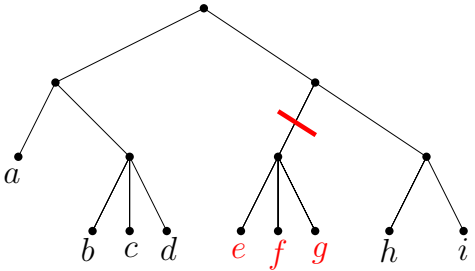

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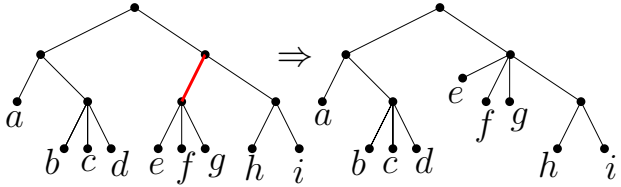

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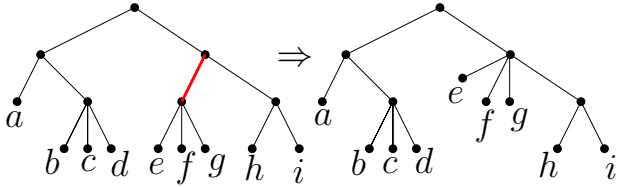

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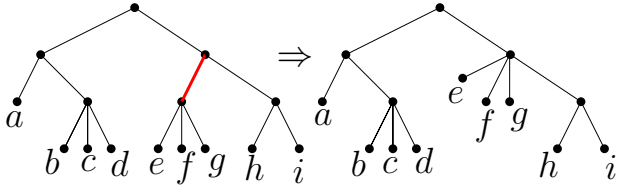

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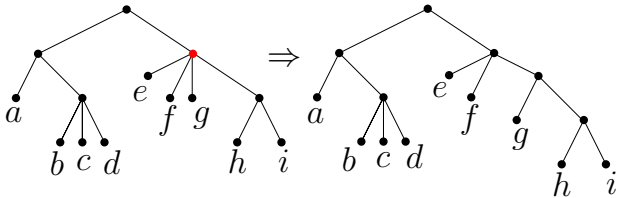

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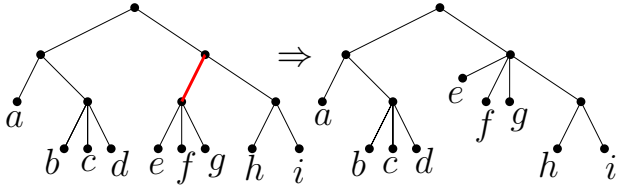

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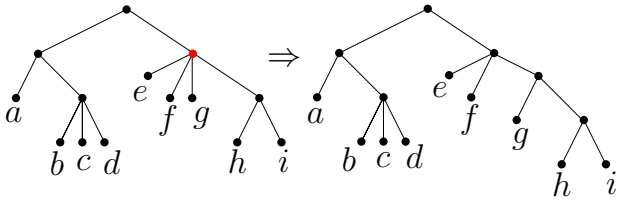

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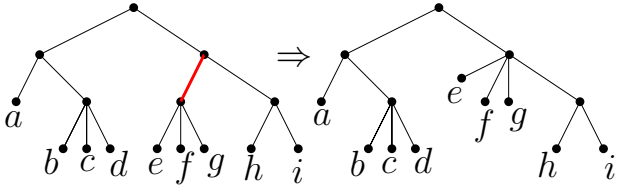

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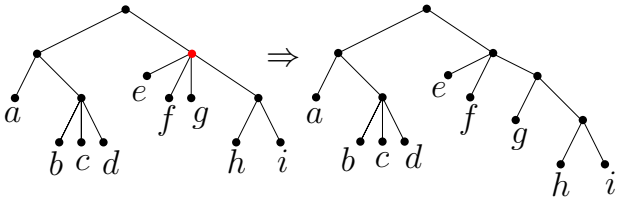

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Dynamic irrelevant vertex technique

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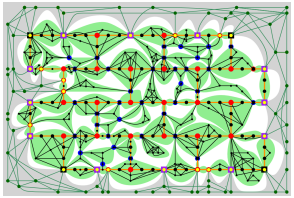

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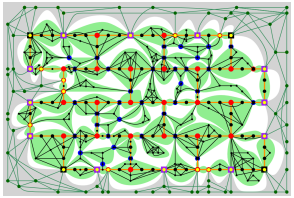

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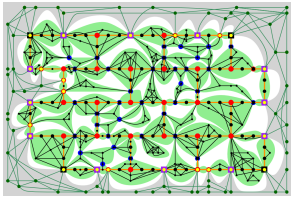

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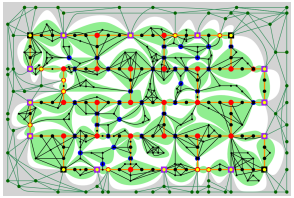

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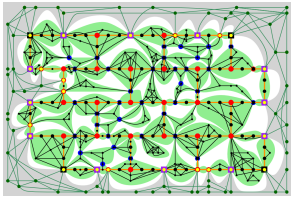

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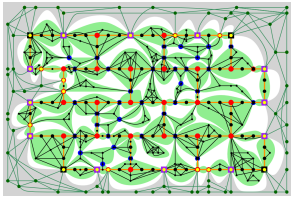

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Dynamic irrelevant vertex technique

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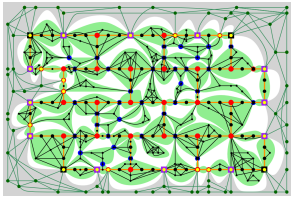

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Dynamic implementation of the irrelevant vertex technique?

- Dynamic k-Disjoint Paths on planar graphs? $f(k) \cdot \log n$, or $f(k) \cdot n^{o(1)}$, or $f(k) \cdot \sqrt{n}$ update time?
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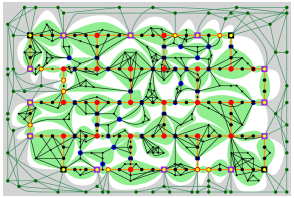

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- Can we separate $f(k) + \mathcal{O}(\log n)$ from $f(k) \cdot \log n$?

Open problems/questions/directions/ideas?

Conclusion

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Thank you!

Dynamic FPT and structural graph theory

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- Minor-closed class with polynomial algorithms [Robertson & Seymour '86]
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Similar role of graph structure in the dynamic setting?

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- Bounded expansion?