Minor Containment and Disjoint Paths in almost-linear time

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based on joint work with Michał Pilipczuk and Giannos Stamoulis
from the University of Warsaw

LIRMM, Montpellier

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- Linear-time algorithms known for some special cases, like planar graphs [Bodlaender 1993], [Reed, Robertson, Schrijver & Seymour 1993]

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   - Need again the graph to be compact

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- a graph G (with m vertices+edges),
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- an X-rooted graph H, and
- an integer p,

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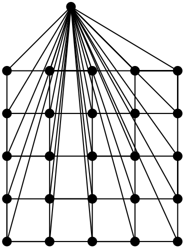

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return in time $\mathcal{O}_{H,|X|}(m^{1+o(1)})$ either

1. whether H is an X-rooted minor of G (and a minor model of H), or
2. a minor model of an apex-grid of order p in $G - X$.

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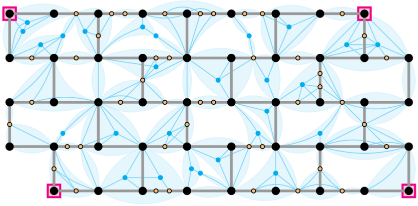

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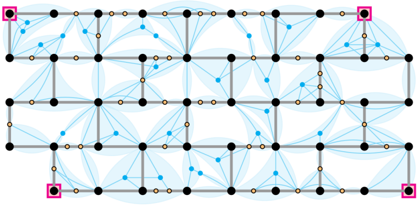

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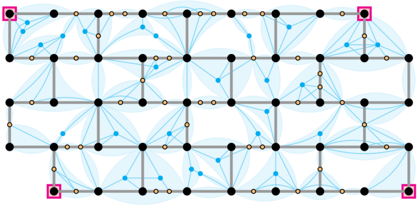
- If $G - X$ is apex-minor-free, then we can assume $Z = X$



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Theorem [K., Majewski, Nadara, Pilipczuk & Sokołowski, FOCS 2023]:

There is data structure that

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- supports edge insertions and deletions in amortized time $f(k) \cdot 2^{\sqrt{\log n} \log \log n} = f(k) \cdot n^{o(1)}$ under the promise that the treewidth of G never exceeds k
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- ...and to deletions of vertices and insertions of isolated vertices

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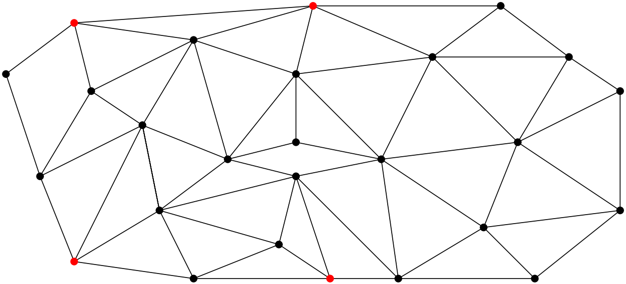

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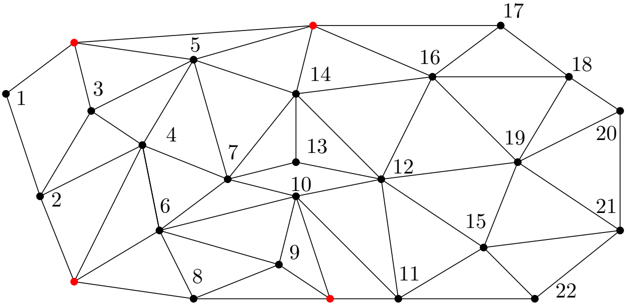

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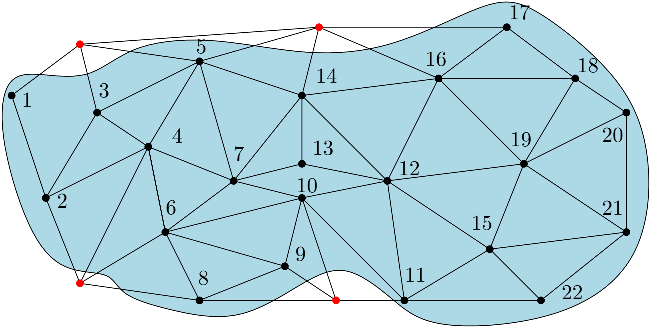

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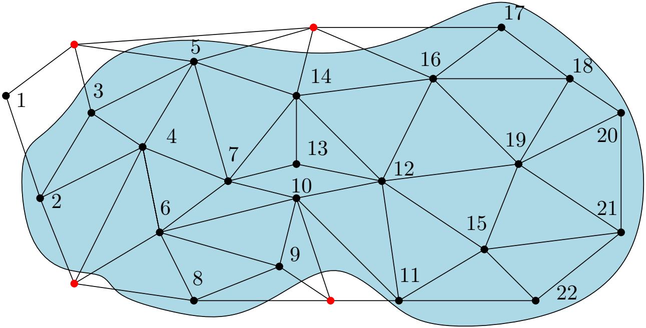

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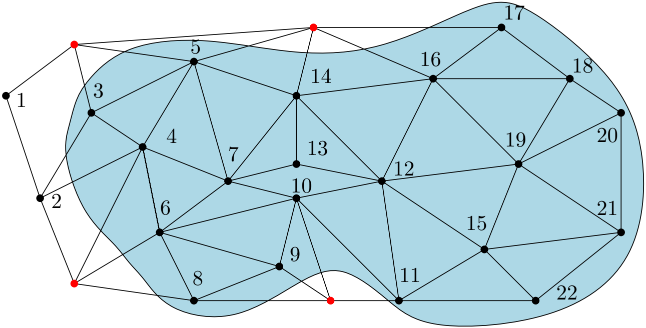

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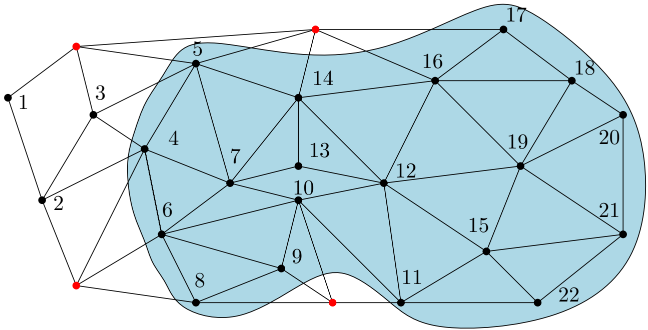

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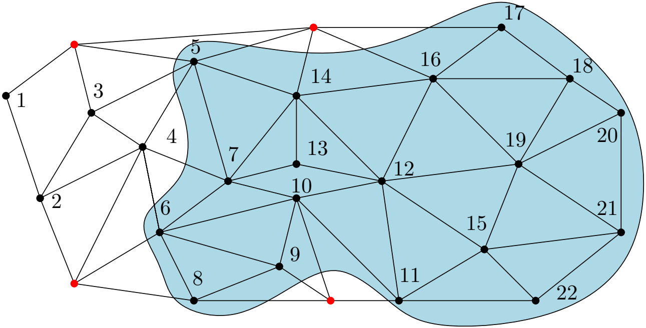

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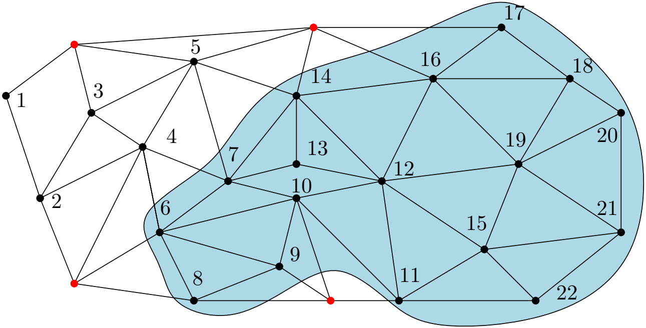

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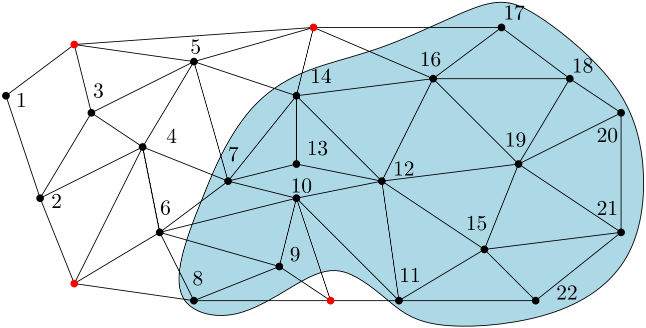

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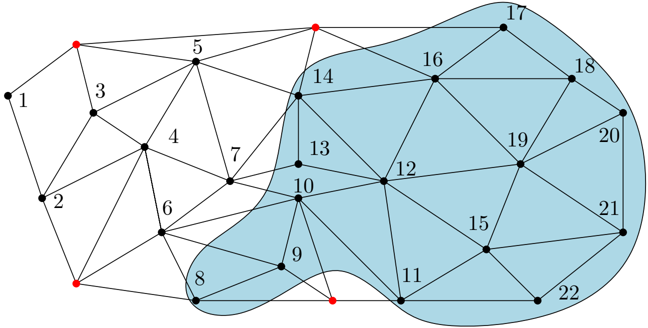

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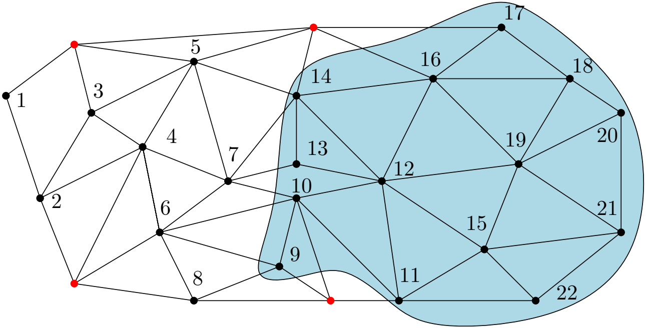

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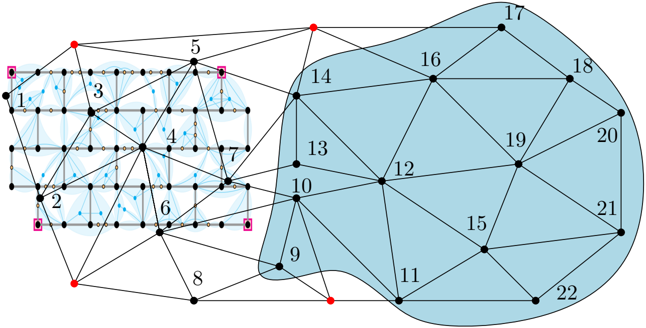

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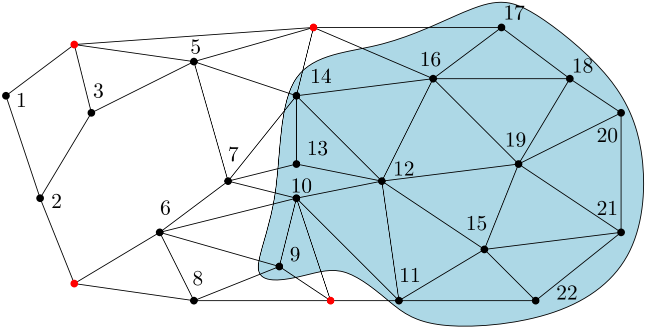

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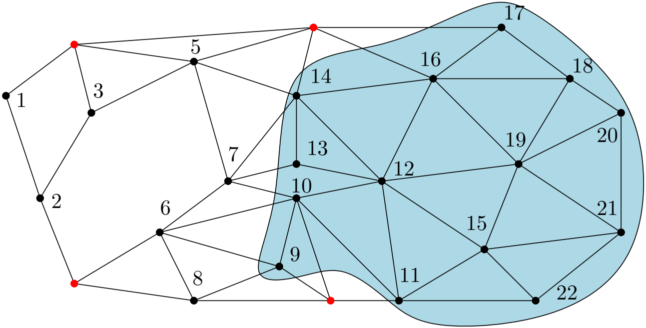

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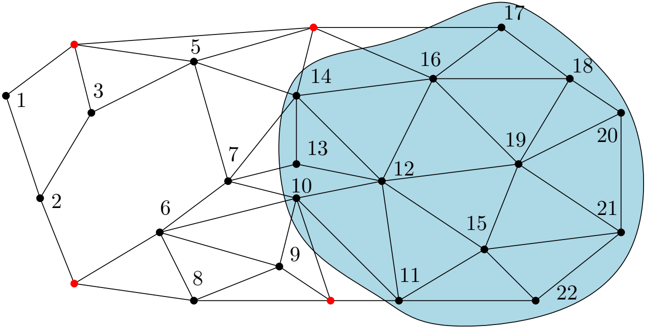

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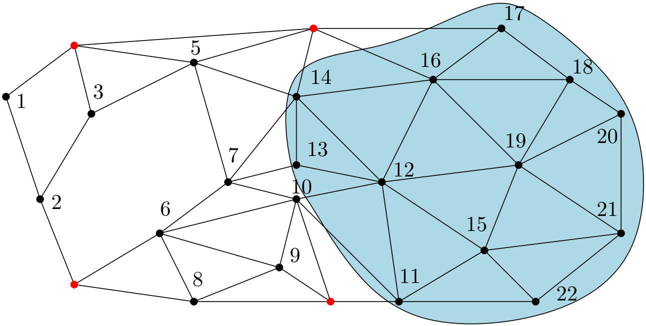

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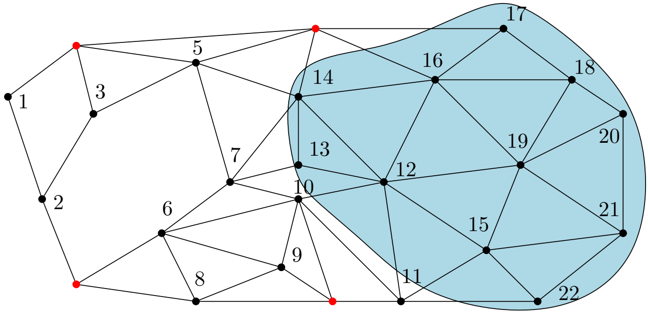

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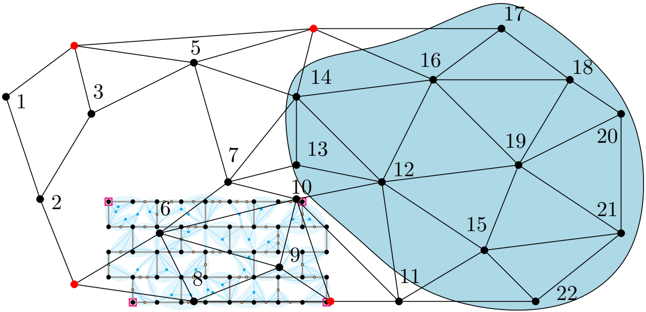

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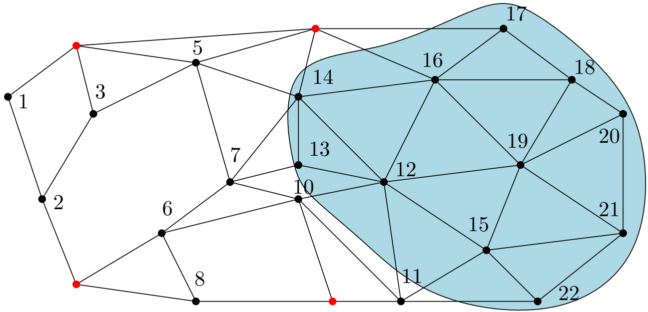

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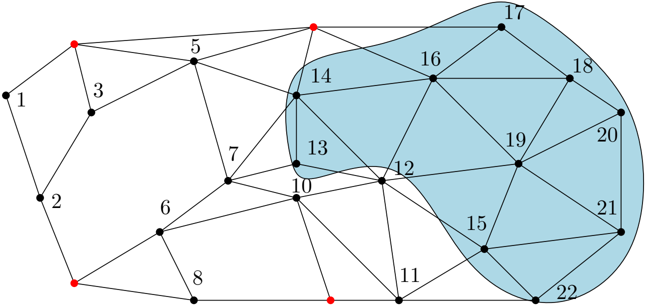

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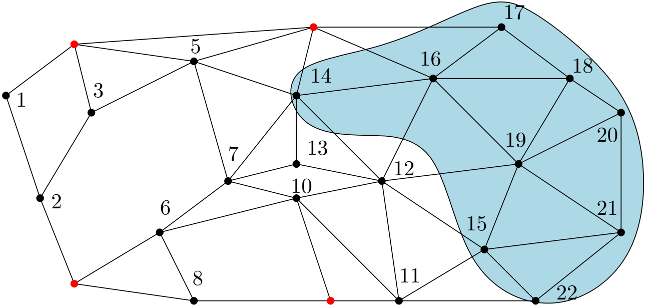

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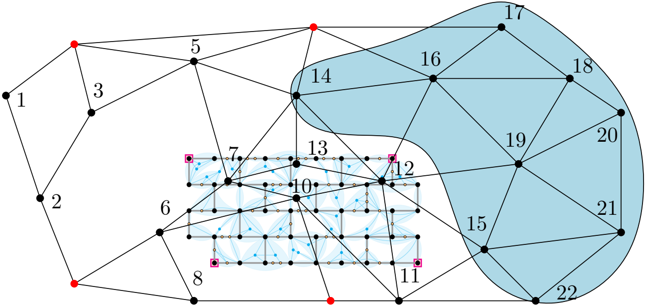

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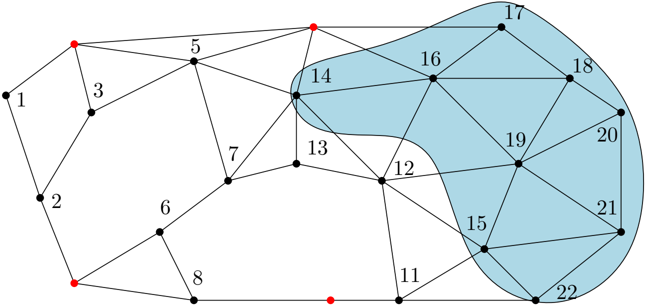

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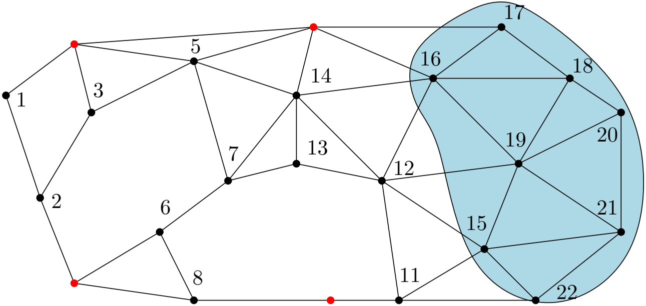

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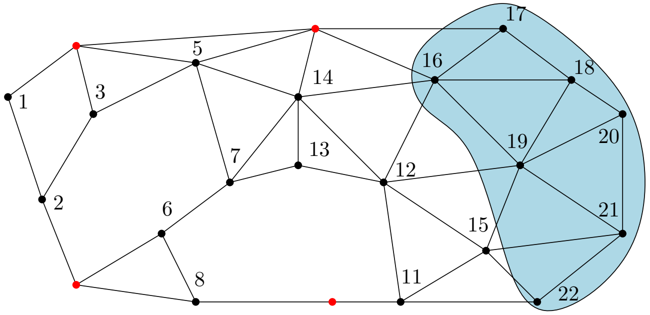

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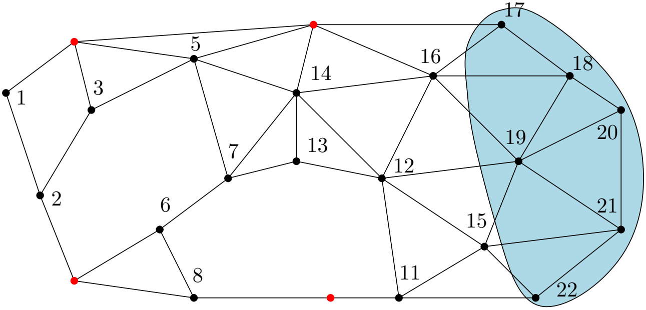

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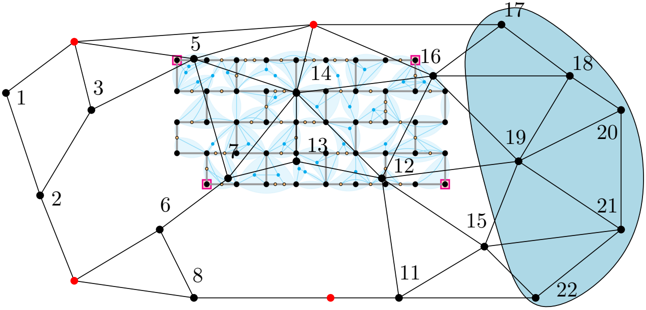

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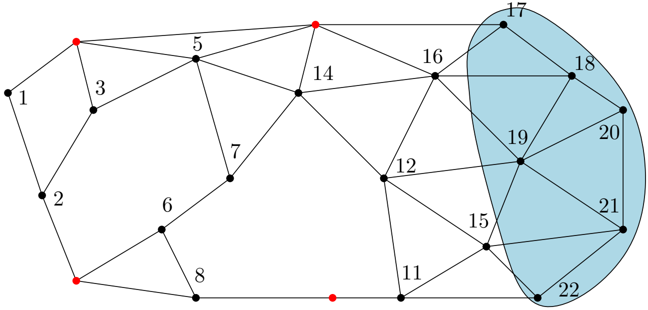

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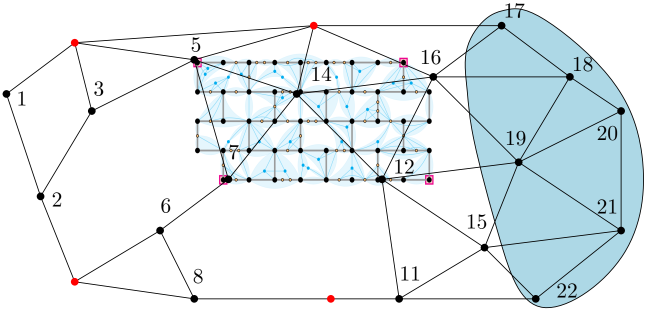

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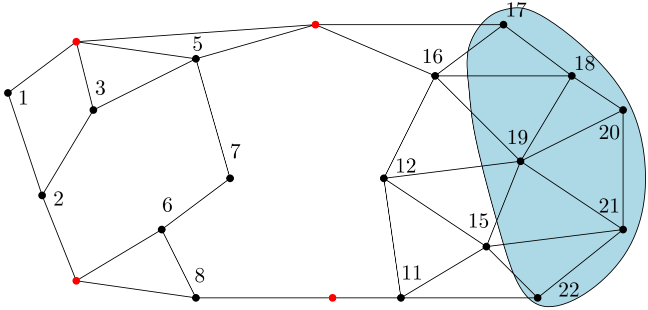

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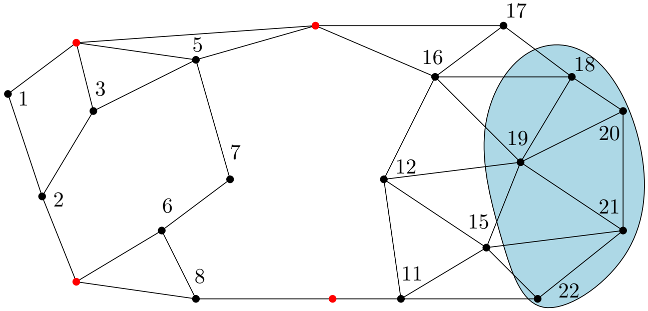

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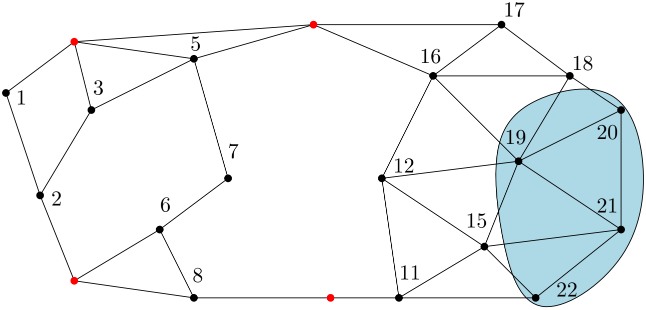

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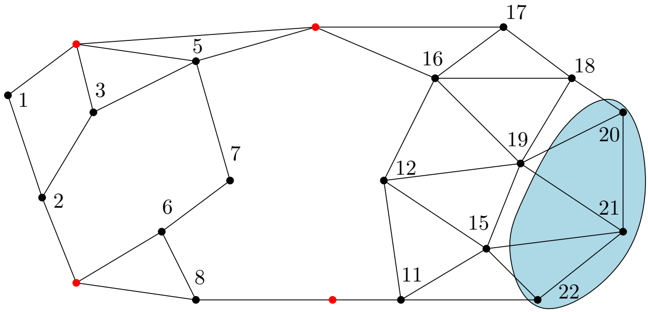

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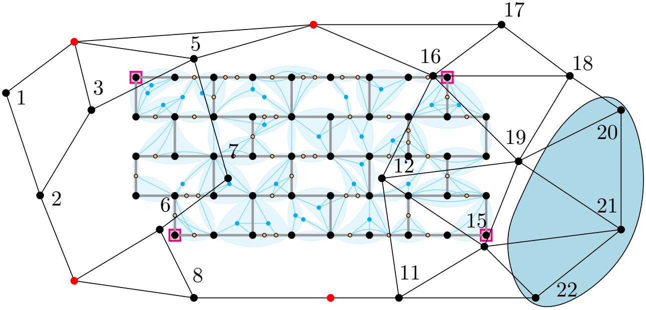

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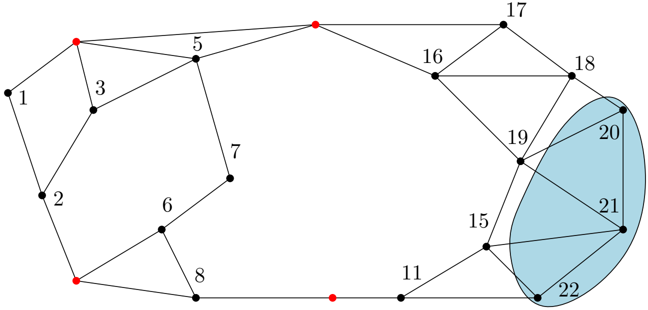

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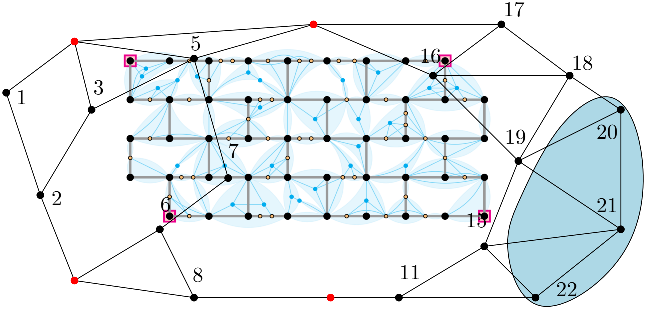

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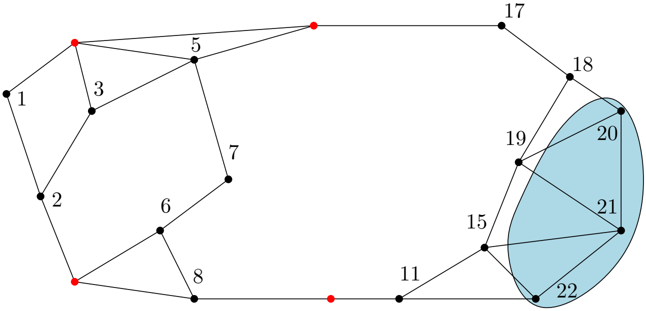

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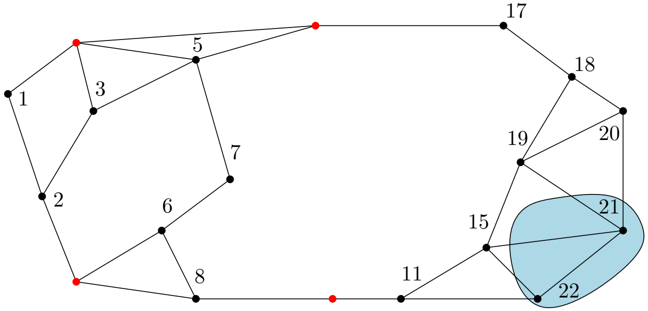

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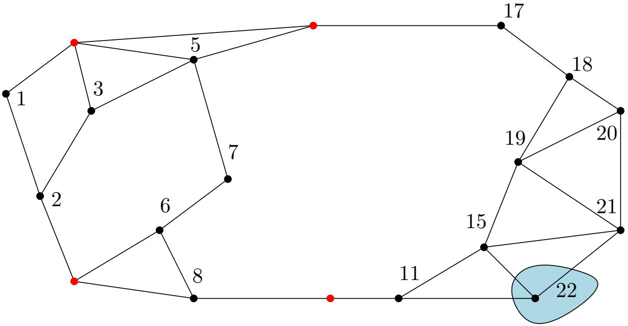

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Compact graphs

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G, X is (k, α)-compact if

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Lemma (Follows from RS-decomposition)

If G excludes K_t, there is $k = f_1(t)$ so that if $|X| \geq k$ and G is (k, α)-compact, then G excludes apex-grid of order $p = f_2(t, \alpha)$ after deletion of $f_3(t)$ vertices.

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Compact clique-minor free $\Leftrightarrow$ almost apex-minor-free

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Recursive scheme:

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   $\Rightarrow$ Recursive understanding to make G, X into (k, α)-compact

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Recursive understanding

Now: X is well-linked and $|X| \geq k$

Want also: There is no connected set $C \subseteq V(G)$ with $|C| \geq \alpha$ and $|N(C)| < k$

- While exists such set C, solve Folio recursively on $(G[N[C]], N(C))$ and replace C by a gadget of size $< \alpha/2$
- Need to do it fast! Two ingredients:

1. Algorithm for finding a collection of non-touching such sets C containing a fraction of $1/\mathcal{O}_{k,\alpha}(\log n)$ of all vertices in such sets
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From general graphs to minor-free

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Need actually a new proof of the lemma to make it more algorithmic

Conclusion

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Thank you!