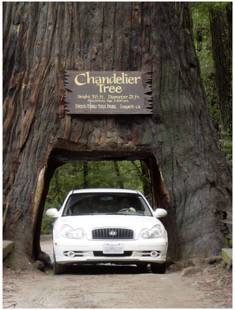
# Computing Treewidth

Tuukka Korhonen

27 February 2025

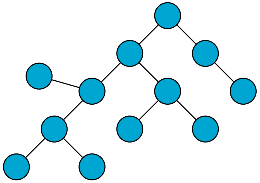
# Plan

1. Introduction to treewidth
2. Background on computing treewidth
3. My work on computing treewidth



# Treewidth: Introduction

- Many algorithmic problems can be solved more efficiently on trees than on general graphs



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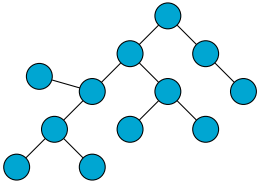

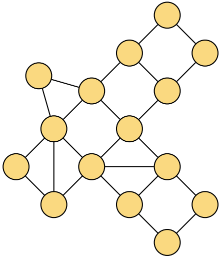

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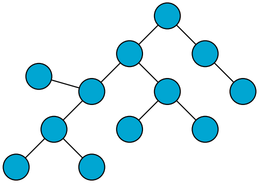

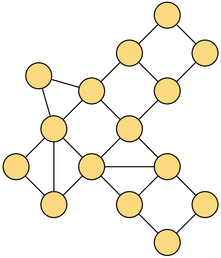

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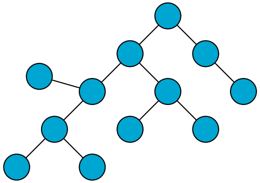

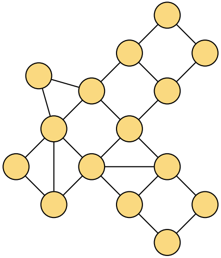

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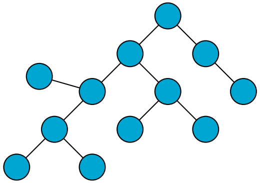

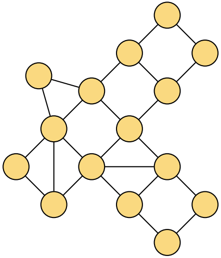

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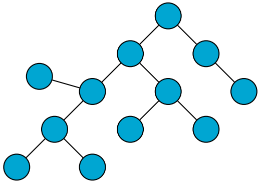
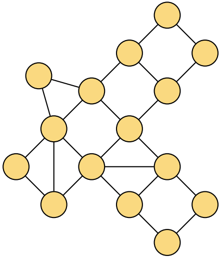
- Applications in graph algorithms, constraint solving, databases, probabilistic inference, simulating quantum computers...





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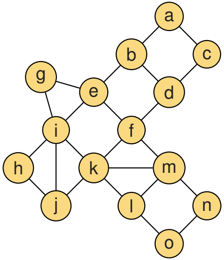


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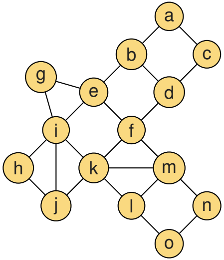


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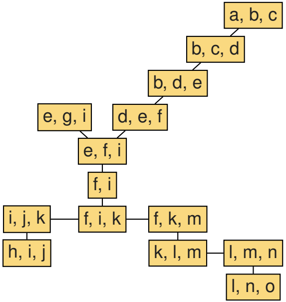


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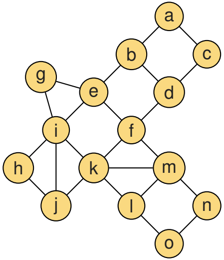


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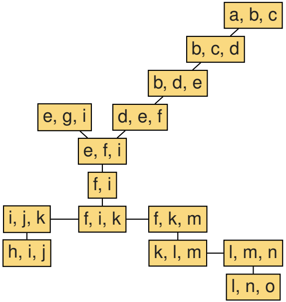


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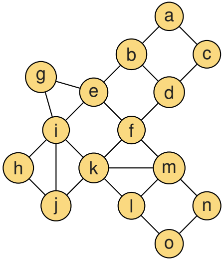


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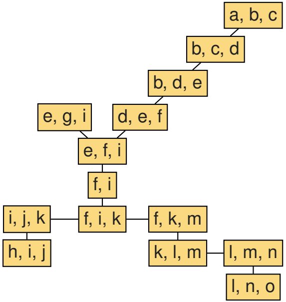


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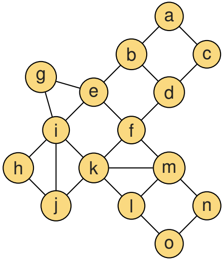


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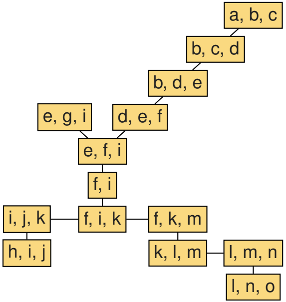


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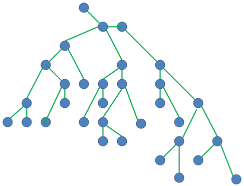
- Treewidth of $G$ = the minimum width of a tree decomposition of $G$

# Treewidth of graphs

Examples of graphs of small treewidth:



Trees (tw $\leq$ 1)

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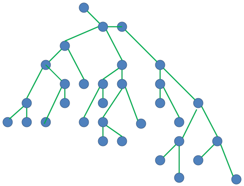

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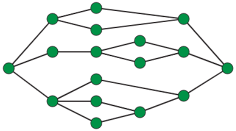

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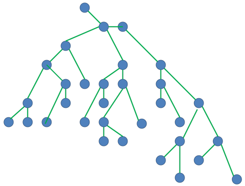

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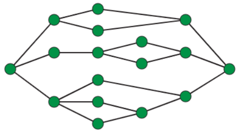

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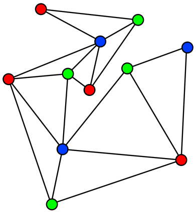

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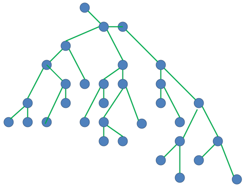

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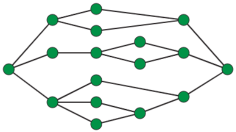

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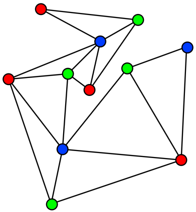

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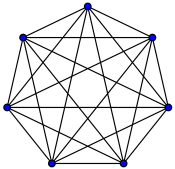

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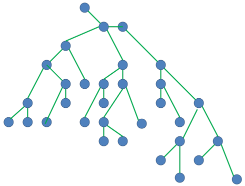

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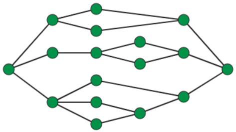

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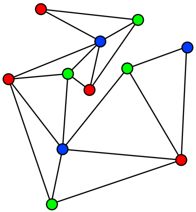

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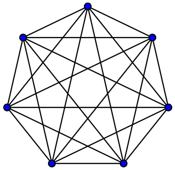

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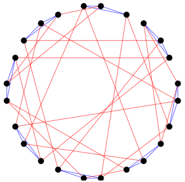

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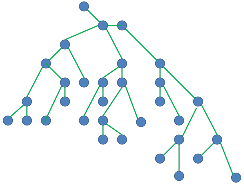

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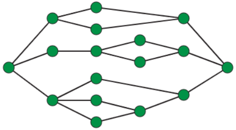

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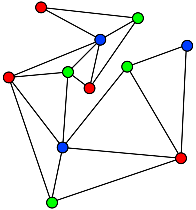

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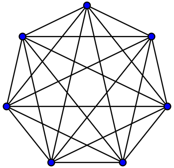

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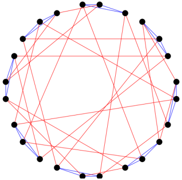

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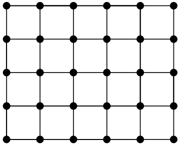

$n \times m$-grids (tw $= \min(n, m)$)

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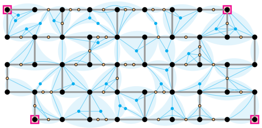

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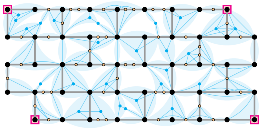

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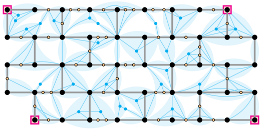

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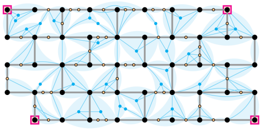

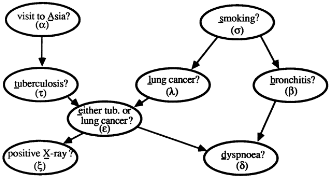

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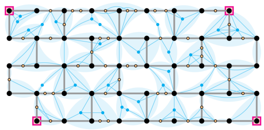

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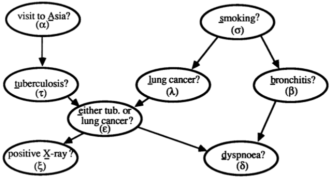
- Mikkel Thorup for compiler optimization, 1997



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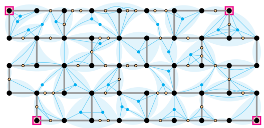

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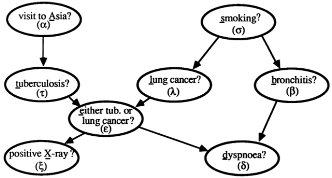
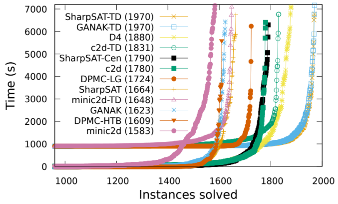



Modern practical applications include at least:

- Probabilistic inference
- Propositional model counting (#SAT)
- Database query evaluation
- Simulating quantum computers



# Treewidth: Example application

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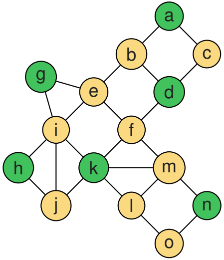

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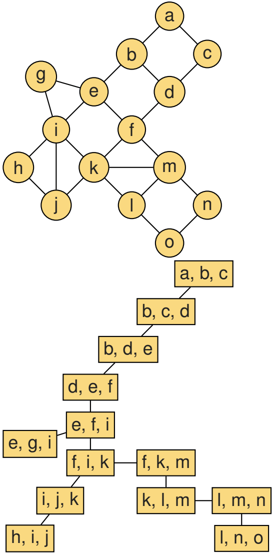

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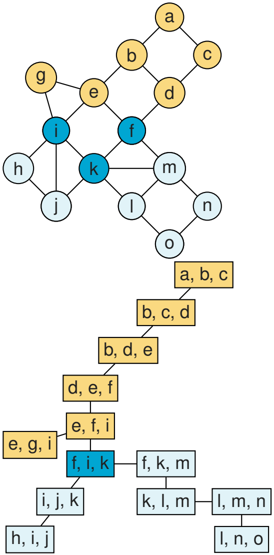


- Example: Solving the maximum independent set problem
- $\mathcal{O}(2^k \cdot n)$ time solution, where $k$ width and $n$ the graph size
- Dynamic programming over states $\text{dp}[t][S]$, where $t$ is a node and $S \subseteq \text{bag}(t)$

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- [Robertson & Seymour, Graph Minors 13, ’87]:
  - 4-approximation algorithm with running time $\mathcal{O}(3^{3k} \cdot n^2)$
  - Introduced the “top-down” approach for computing tree decompositions

# The Robertson-Seymour top-down approach

Graph

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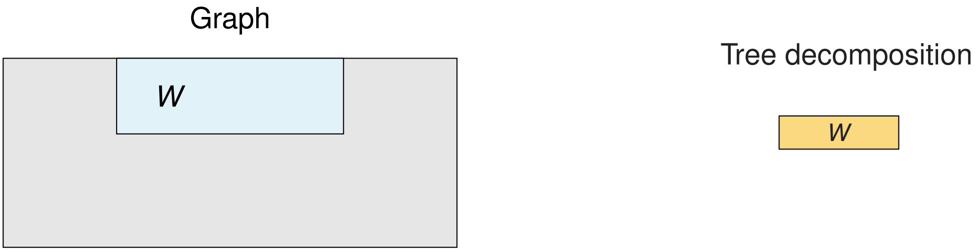

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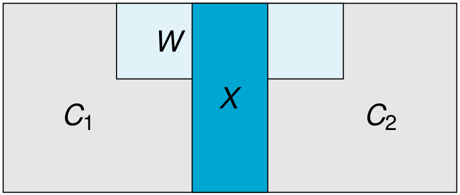


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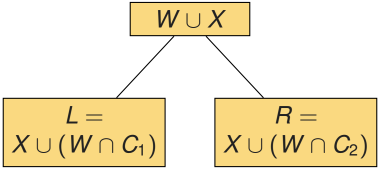
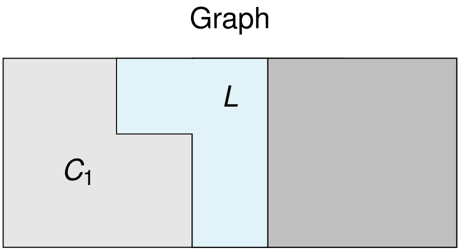
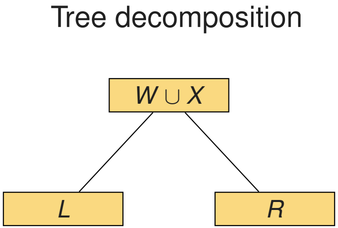



Balanced separator $X$ with components $C_1$ and $C_2$

# The Robertson-Seymour top-down approach



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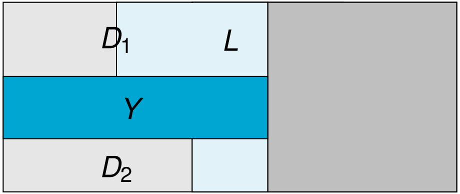


Balanced separator $Y$ with components $D_1$ and $D_2$

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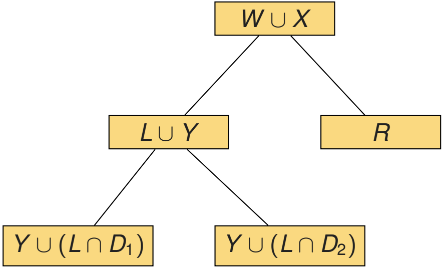

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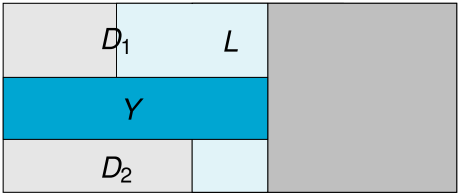


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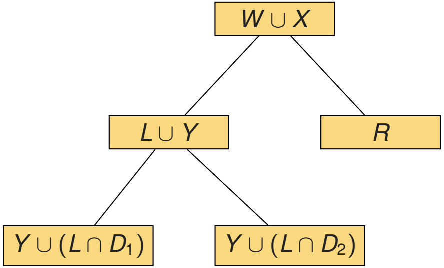
Continue recursively...

Tree decomposition



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- Barrier at approximation ratio 3
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- A completely new approach
  - Inspired by the proofs of [Thomas '90] and [Bellenbaum & Diestel '02] on "lean tree decompositions"

# The 2-approximation algorithm

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By a self-reduction technique of [Bodlaender '96] we can focus on giving an improver algorithm:

**Input:** An graph $G$ and a tree decomposition $T$ of $G$ of width $w$

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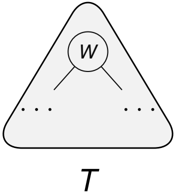
   - Efficient implementation by amortized analysis of the improvements and dynamic programming over the tree decomposition

# The improvement operation

- Let $W$ be the largest bag



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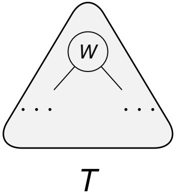

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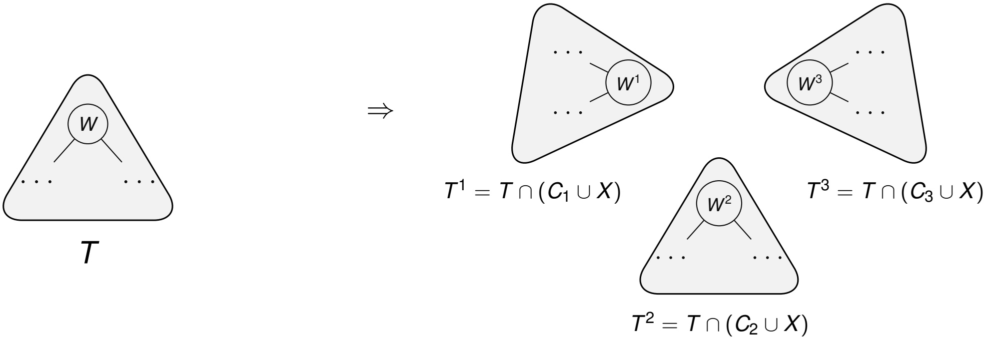

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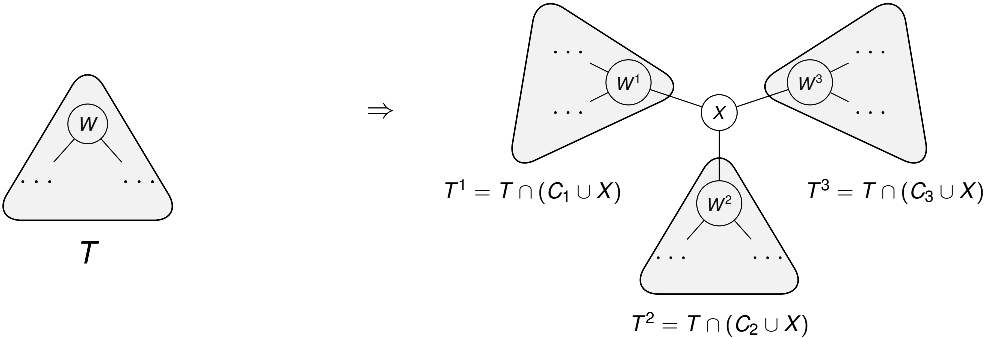

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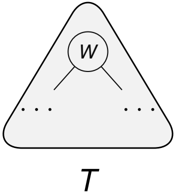


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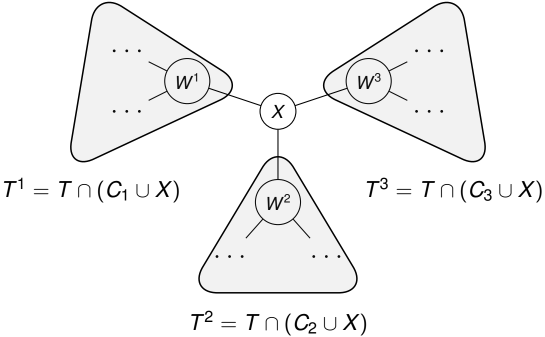


Except that vertices in $X$ may violate the connectedness condition

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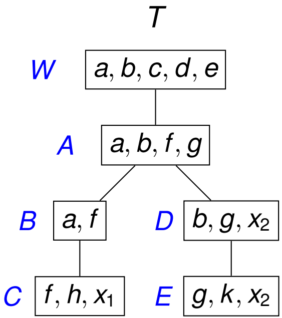

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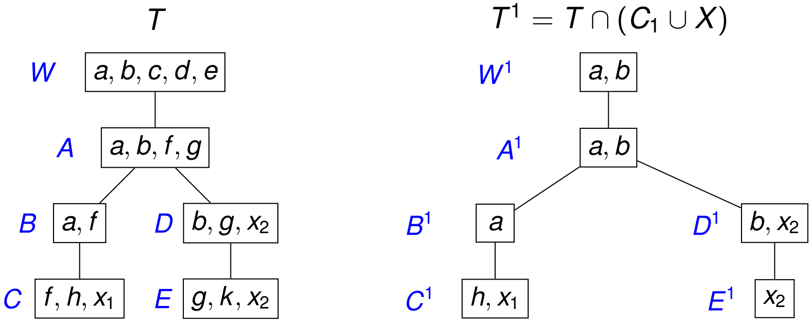

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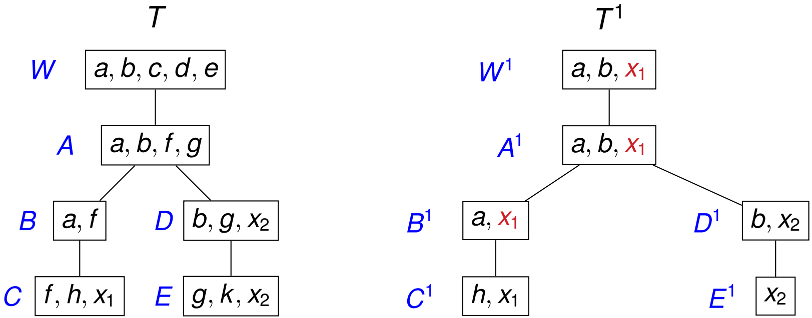


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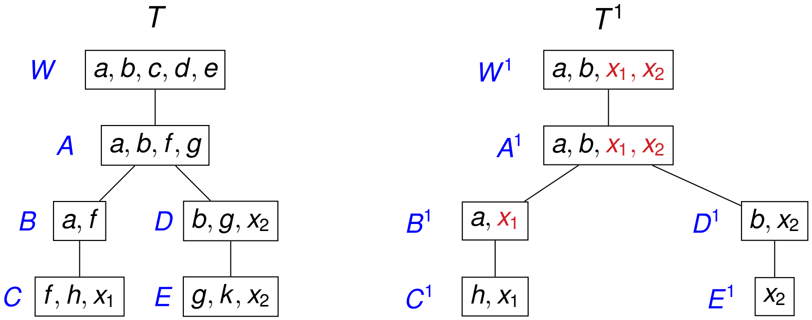


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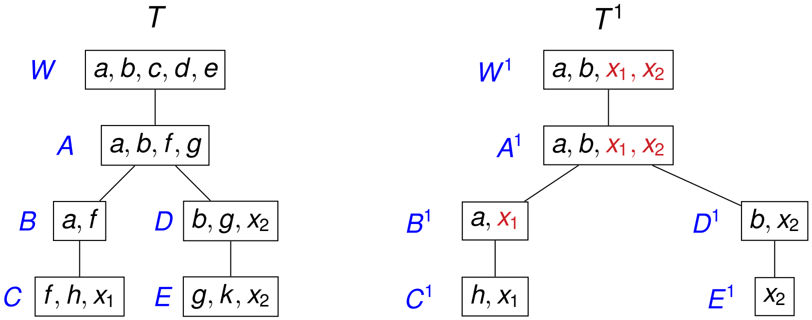


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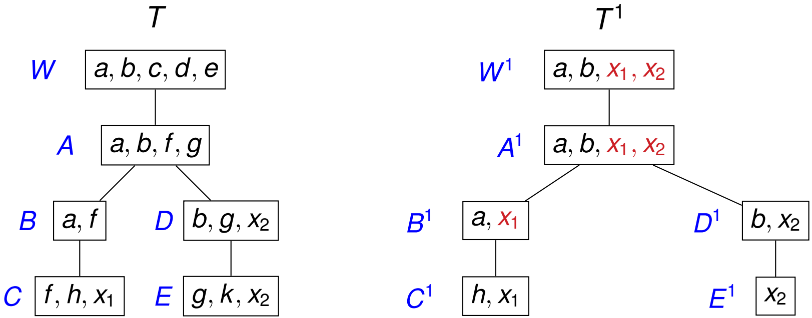


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$\Rightarrow$ The whole construction satisfies the connectedness condition

# The main insight

**Definition (Good separation)**

A separation $(X, C_1, C_2, C_3)$ is a *good separation* if (1) $|X \cup (W \cap C_i)| < |W|$ for all $i$, and (2) among those, we minimize $|X|$.

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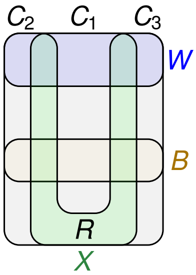

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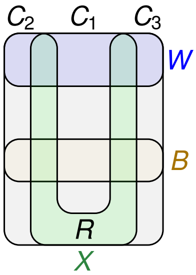

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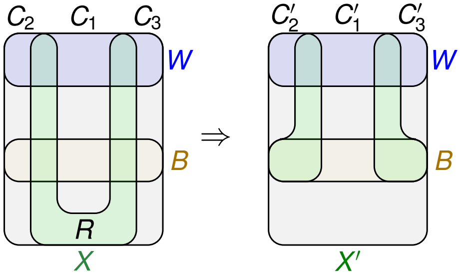
- Take a separation with $X' = (X \setminus R) \cup (B \cap (C_2 \cup C_3))$



# The main insight

## Definition (Good separation)

A separation $(X, C_1, C_2, C_3)$ is a *good separation* if (1) $|X \cup (W \cap C_i)| < |W|$ for all $i$, and (2) among those, we minimize $|X|$.

## Lemma

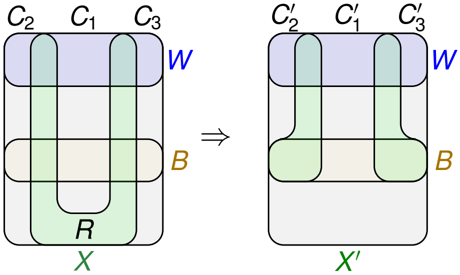
If $(X, C_1, C_2, C_3)$ is a good separation, then $|B^i| \leq |B|$ for all bags $B$ and all $i$

- Suppose that $i = 1$ and $|B^1| > |B|$

$\Rightarrow$ $|R| > |B \cap (C_2 \cup C_3)|$

- Take a separation with $X' = (X \setminus R) \cup (B \cap (C_2 \cup C_3))$

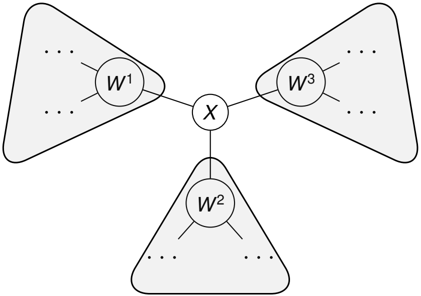
$\Rightarrow$ $|X'| < |X|$ so this contradicts the minimality



# Outlook

We have shown:

- Root bag $W$ replaced by four smaller bags, $W^1$, $W^2$, $W^3$, and $X$
- Width did not increase



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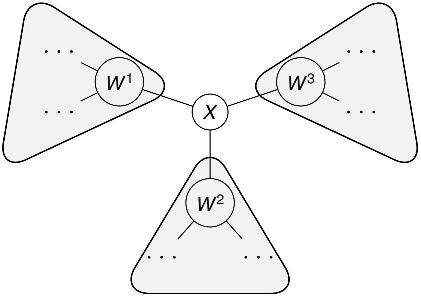

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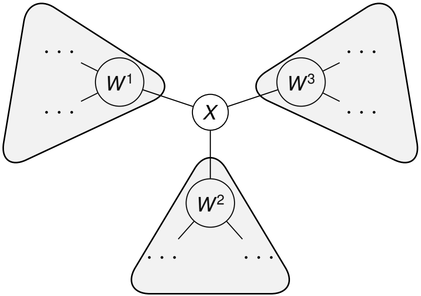

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Should also show:

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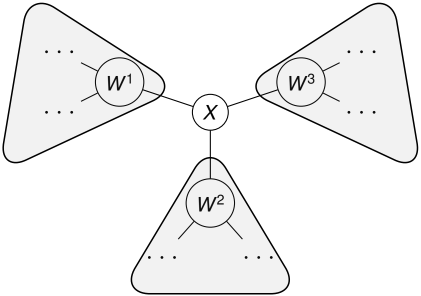
- Good separations can be found efficiently
- $n$ iterations of improvements can be implemented in total $2^{\mathcal{O}(w)} n$ time



# Final remarks

# Follow-up work

Treewidth 2-approximation
[Korhonen, FOCS'21]

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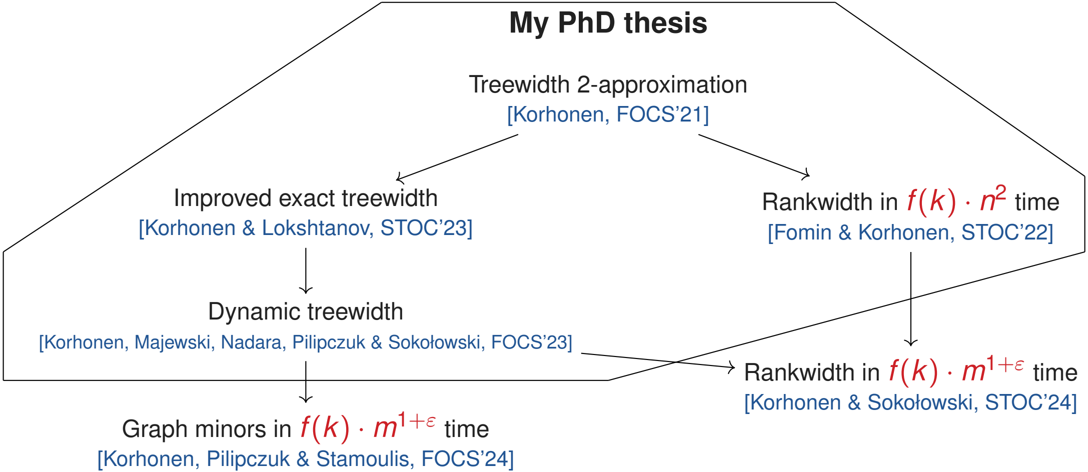
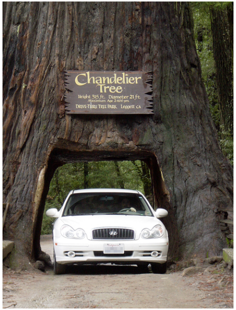


# Conclusion

- Treewidth has applications in many areas of computer science



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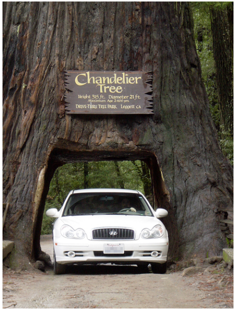
- My PhD thesis: New paradigm in treewidth computing



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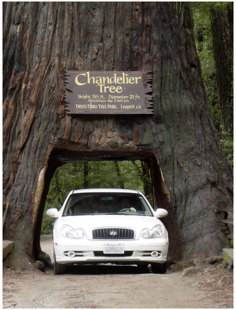

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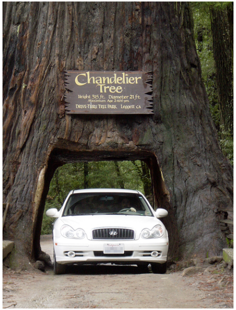

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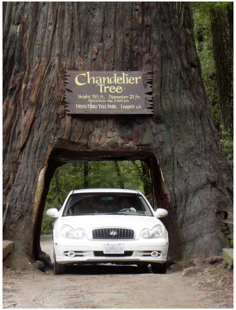


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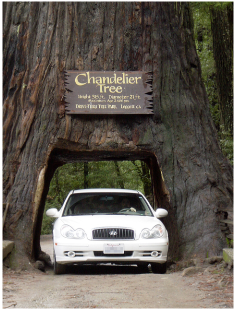


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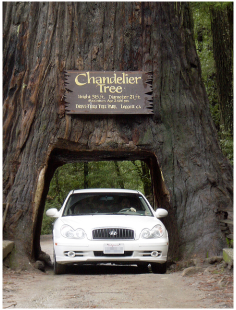


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Thank you!