Lower Bounds on Dynamic Programming for Maximum Weight Independent Set

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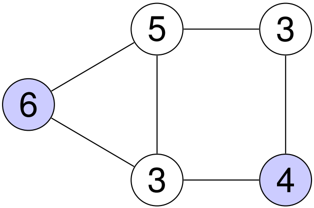
Maximum Weight Independent Set

Given a vertex-weighted graph, determine the weight of a maximum weight independent set (MWIS)



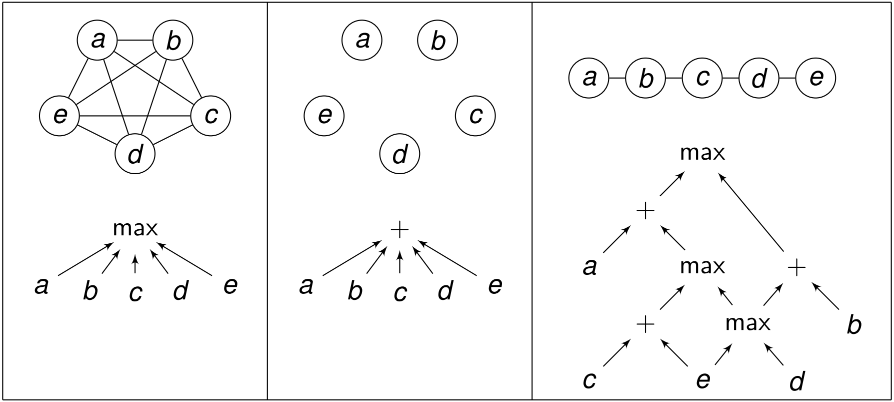
$\Rightarrow$ 10

MWIS-Circuits

MWIS-circuit of a graph G is a circuit with

- Vertex weights as inputs
- $\max$ and $+$ operations as gates (tropical circuit)
- Computes the weight of MWIS of G for any assignment of weights

Examples:



Why Do I Care?

Many known algorithms for MWIS actually build MWIS-circuits

- Dynamic programming on tree decompositions
- Chordal graphs (and almost-chordal graphs)
- Clique-width
- Algorithms based on potential maximal cliques (and their generalizations)
- Branching algorithms

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Lower bounds for MWIS-circuits $\Rightarrow$ lower bounds for algorithmic techniques

Theorem 1

Let

- $tw(G)$ – the treewidth of a graph G
- $d(G)$ – the maximum degree of a graph G

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For every graph G, the size of any MWIS-circuit of G is $2^{\Omega(tw(G)/d(G))}$

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Note: For $d(G) = O(1)$, a $2^{\Omega(tw(G))}$ lower bound follows from earlier work on DNNF-compilation [Amarilli, Capelli, Monet and Senellart, 2020]

Theorem 2

Graph H is an *induced minor* of G if we can obtain H from G by vertex deletions and edge contractions. (edge deletions **not** allowed)

Definition (Bounded-degree induced minor width)

Let $bdimw(G)$ denote the maximum of $tw(H)$, where H is an induced minor of G with maximum degree $O(1)$.

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$\Rightarrow$ Matching upper and lower bounds when $bdimw(G) = \Omega(tw(G))$.

- Bounded-degree graphs

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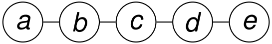
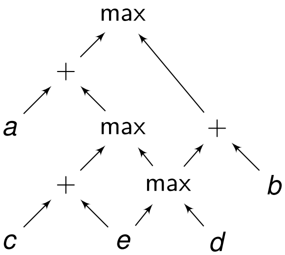
- Bounded-degree graphs
- Planar graphs
- Bounded-genus graphs
- Graphs excluding some fixed minor

MWIS-formulas

MWIS-formula is an MWIS-circuit whose underlying graph is a tree



MWIS-circuit



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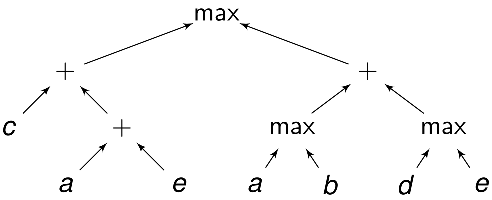

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Again,

1. $2^{O(td(G))}|V(G)|$ size MWIS-formulas known, so instance-optimality for $d(G) = O(1)$
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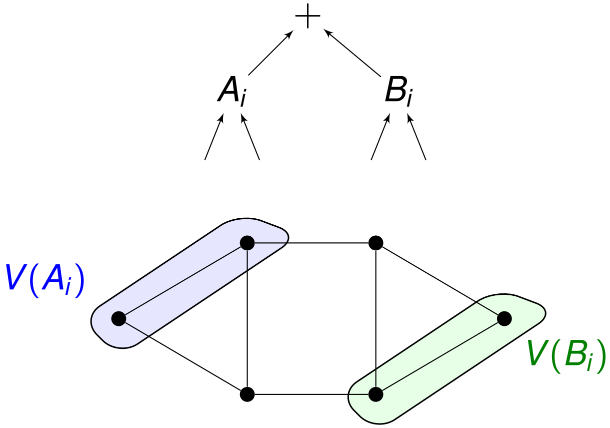
However, no $2^{O(td(G))}|V(G)|^{O(1)}$ time constant-factor approximation algorithm for treedepth known, so the instance-optimality is “non-uniform”

Proof idea of Theorem 1

Consider a $+$ gate computing $A_i + B_i$, where A_i and B_i are gates



Let $V(A_i)$ be vertices that have a path to A_i and $V(B_i)$ vertices that have a path to B_i.

$V(A_i)$ and $V(B_i)$ must be disjoint, without edges in between.

Circuit Decomposition

- Treewidth $tw(G)$ guarantees a vertex subset $X \subseteq V(G)$ s.t. every balanced separator of X has size $\Omega(tw(G))$.

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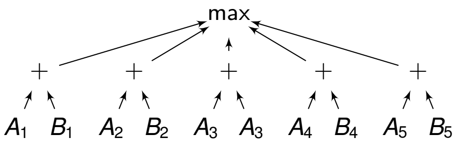
- Circuit of form $\max(A_1 + B_1, A_2 + B_2, A_3 + B_3, \ldots, A_\tau + B_\tau)$, where $|V(A_i) \cap X| \leq 2|X|/3$ and $|V(B_i) \cap X| \leq 2|X|/3$, and τ the number of gates in the original circuit



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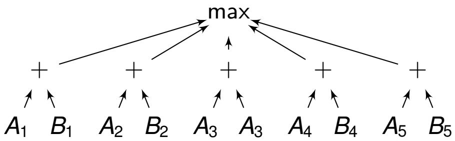

- Let $S_i = V(G) \setminus (V(A_i) \cup V(B_i))$, note that $|S_i| = \Omega(tw(G))$
- We can fool the circuit by constructing an IS that intersects every S_i

Fooling the Circuit

Given τ vertex subsets $S_1, S_2, \ldots, S_\tau$ of size $|S_i| \geq k = \Omega(tw(G))$, can we construct an independent set that intersects them all?

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Theorem

Yes, if $\tau \leq e^{k/(6d(G))}$

Proof idea:
Apply Lopsided Lovász Local Lemma, showing that when choosing each vertex with probability $1/(2d(G))$, we get such IS with non-zero probability

Other proof ideas

- Theorem 2. ($2^{\Omega(bdimw(G))}$ lower bound for MWIS-circuits):
  - Map an unbreakable set $X \subseteq V(H)$ of the induced minor H to a set $X' \subseteq V(G)$, and then decompose the circuit w.r.t. X'.
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  - A different application of the local lemma that works for $d(H) = O(1)$.
- Theorem 3. ($2^{\Omega(td(G)/d(G))}$ lower bound for MWIS-formulas):
  - Ad-hoc extraction of $O(\tau)$ vertex sets of size $\Omega(td(G))$ such that an independent set that intersects them all fools the circuit
  - Same application of the local lemma as in Theorem 1.

Future work

1. Prove similar lower bounds for other graph problems (min weight vertex cover, max weight matching, . . .)
2. Prove analogue of Theorem 2 for MWIS-formulas
3. Go deeper on MWIS-circuits

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2. Prove analogue of Theorem 2 for MWIS-formulas

3. Go deeper on MWIS-circuits
   - Does a $2^{\Omega(tw(G))}$ lower bound hold for all bounded-degeneracy graphs?
   - Are there MWIS-circuits of size FPT or XP parameterized by $bdimw(G)$? (single-exponential FPT unlikely)

The end

Thank you for your attention!

For any questions/comments, please attend ICALP session 2A or send me email
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Bibliography

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