Fast FPT-Approximation of Branchwidth

Fedor V. Fomin, Tuukka Korhonen

University of Bergen, Norway

Parametrized complexity and discrete optimization
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Branchwidth

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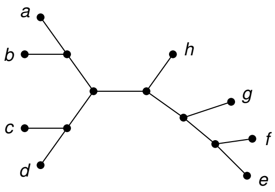

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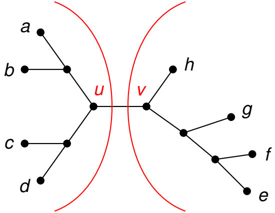

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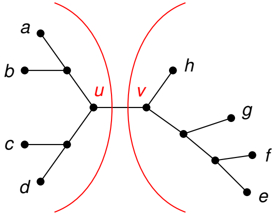

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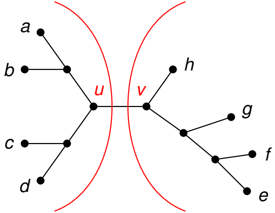

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Connectivity functions

- Function $f : 2^V \to \mathbb{Z}_{\geq 0}$ is a connectivity function if for any $A, B \subseteq V$:
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- Cut-rank in a graph G:
  - For any vertex set $A \subseteq V(G)$ let $\mathrm{cutrk}_G(A)$ be the GF(2)-rank of the $|A| \times |\overline{A}|$ matrix representing edges between A and $\overline{A}$.
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Techniques for rankwidth

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Input: Augmented rank decomposition of G of width k
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- Algorithmic framework:
  - Direct computation of refinements by dynamic programming $\rightarrow 2^{2^{\mathcal{O}(k)}} n^2$ time
  - Amortization techniques exploiting combinatorial results $\rightarrow 2^{2^{\mathcal{O}(k)}} n$ time

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- Our structural result:
  - An edge uv of the decomposition is *heavy* if $f(uv) = k$
  - If $k > 2\texttt{bw}(f)$, then a **refinement operation** can be applied, which decreases the number of heavy edges and does not increase the width

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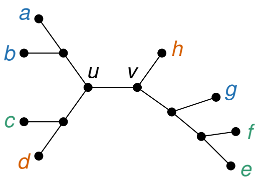

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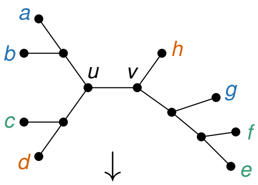

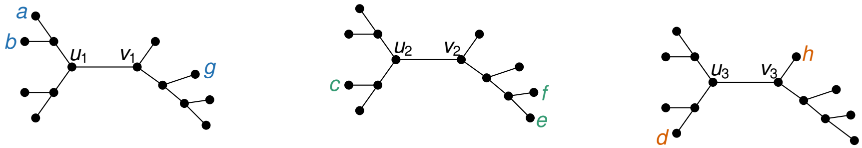

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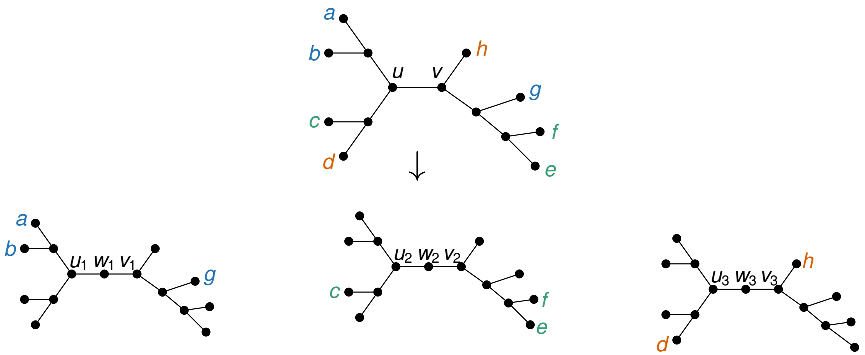

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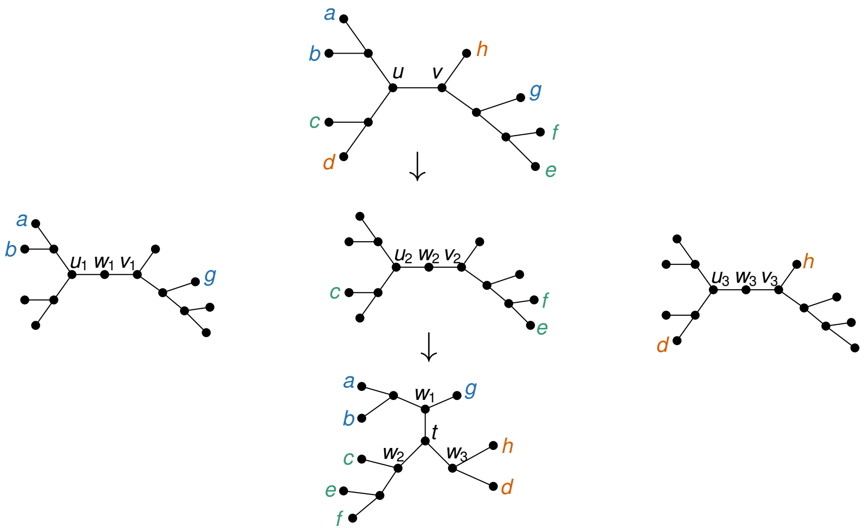

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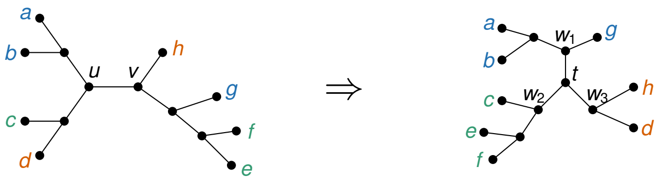

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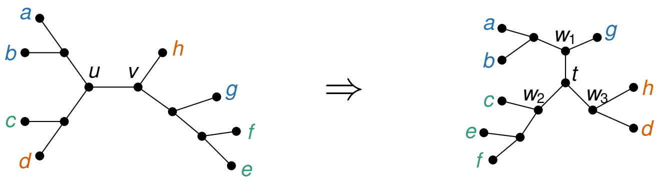

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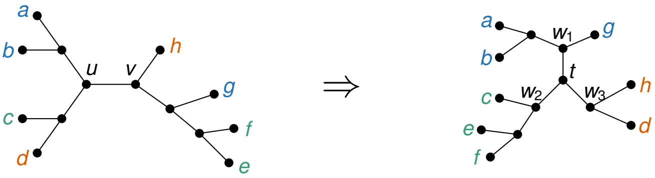

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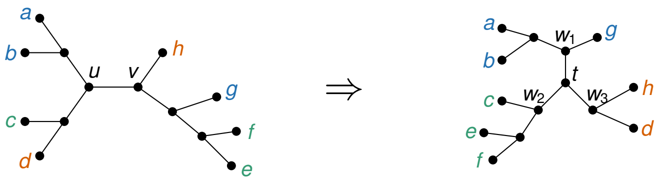

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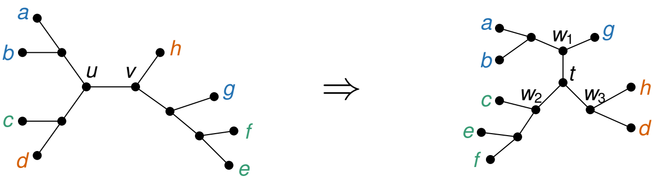

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- Observation 2: For each i, there will be edges corresponding to $(C_i \cap W, \overline{C_i \cap W})$ and $(C_i \cap \overline{W}, \overline{C_i \cap \overline{W}})$

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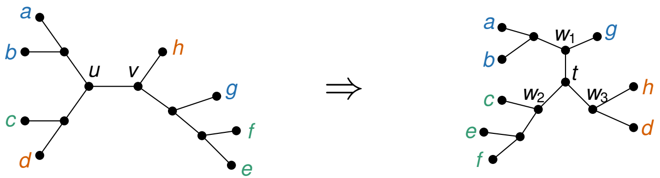

- Let $(W, \overline{W}) = (\{a, b, c, d\}, \{e, f, g, h\})$ be the cut of uv
- Combination of Observation 1 and 2:
  - The widths of edges "near the center" will be $f(C_i)$, $f(C_i \cap W)$, and $f(C_i \cap \overline{W})$ for each i

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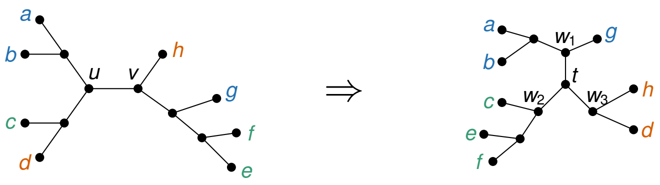

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For any set $W \subseteq V$ with $f(W) > 2\mathtt{bw}(f)$ there exists tripartition (C_1, C_2, C_3) of V so that for each i it holds that $f(C_i) < f(W)/2$, $f(C_i \cap W) < f(W)$, and $f(C_i \cap \overline{W}) < f(W)$.

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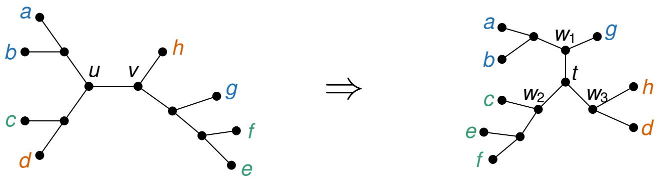

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$\Longrightarrow$ If $f(uv) > 2\mathtt{bw}(f)$, there exists refinement with uv that locally improves T

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- Recall: If $f(uv) > 2\mathtt{bw}(f)$, then W-improvement exists

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- Let $uv \in E(T)$, $(W, \overline{W})$ the cut of uv, and $f(uv) = k$
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If there exists a W-improvement, then there exists a W-improvement (C_1, C_2, C_3) so that refinement with (uv, C_1, C_2, C_3) does not increase width and decreases number of edges of width k.

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  - i.e., primarily minimize $\max_i f(C_i)$, secondarily minimize number of non-empty C_i, tertiarily...

First Algorithm

- Now, we have a following meta-algorithm for connectivity functions that allow efficient dynamic programming

1. Let T have width k, select edge uv with $f(uv) = k$
2. Root T at uv
3. Use dynamic programming on T to find a W-improvement optimizing the criteria or conclude $k \leq 2\mathtt{bw}(f)$ if no W-improvement found
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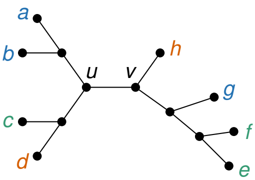
- Too slow! Applications require $t(k) \cdot n$

Amortization technique

Example with $(r, C_1, C_2, C_3) = (uv, \{a, b, g\}, \{c, e, f\}, \{d, h\})$



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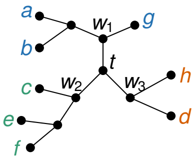

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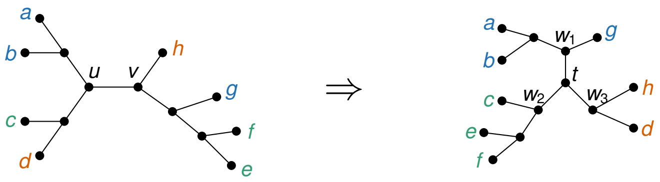

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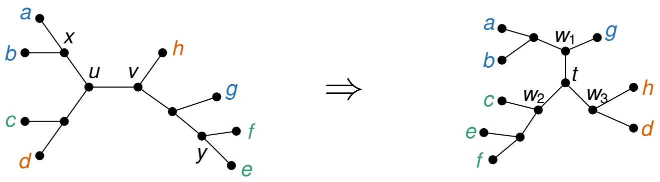

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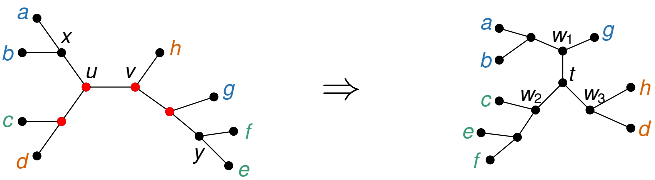

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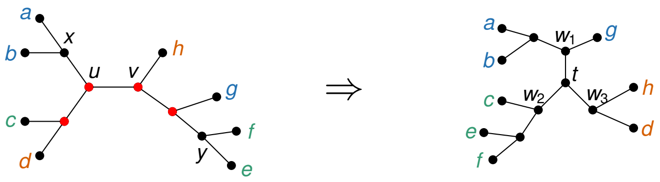

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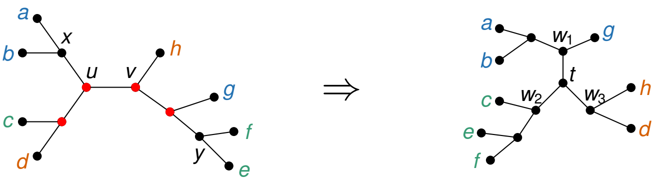

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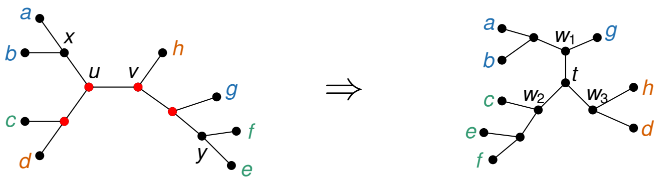

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  - Solves the open problem of breaking the n^3 barrier for rankwidth
- Open problem: Is there a $f(k)(n+m)^{1.9}$ time $g(k)$-approximation algorithm for rankwidth?