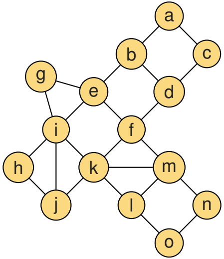
# Dynamic Treewidth in Logarithmic Time

Tuukka Korhonen

FOCS 2025

16 December 2025

# Treewidth




Graph $G$

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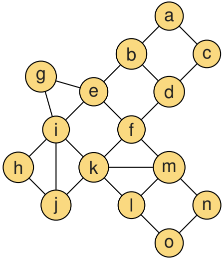


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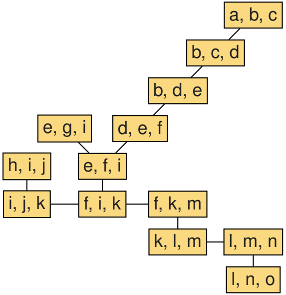


A tree decomposition of *G*

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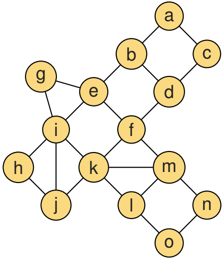


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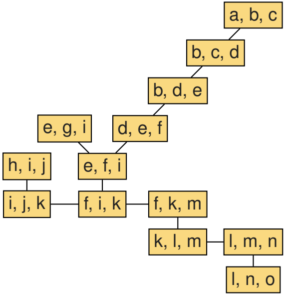


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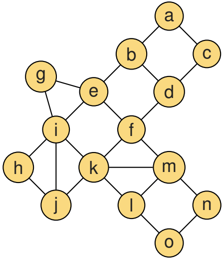


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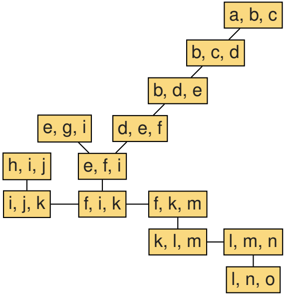


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1. Every vertex should be in a bag
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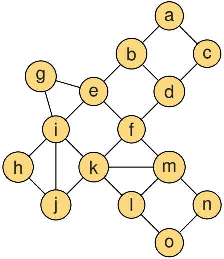


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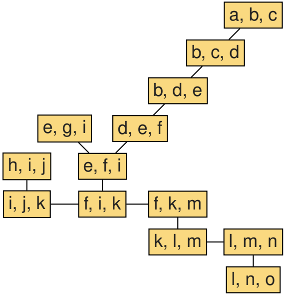


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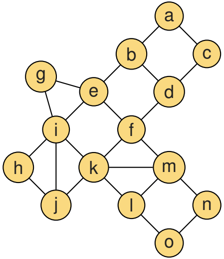


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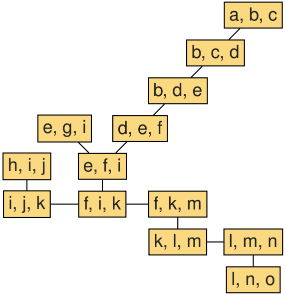


A tree decomposition of $G$

Width = 2

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4. Width = maximum bag size $-1$

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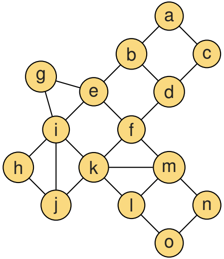


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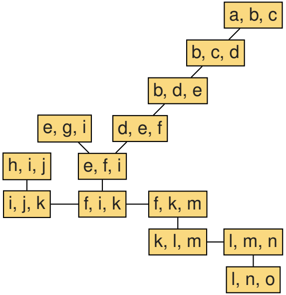


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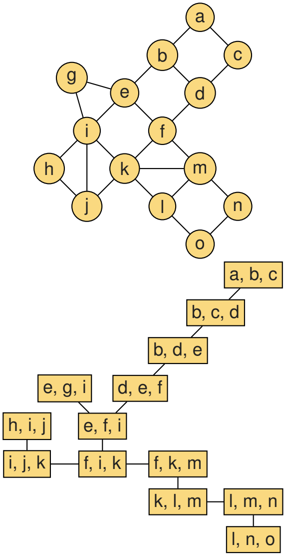
4. Width = maximum bag size $-1$
5. Treewidth of $G$ = minimum width of tree decomposition of $G$

[Robertson & Seymour'84, Arnborg & Proskurowski'89, Bertele & Brioschi'72, Halin'76]

# Why treewidth?

- Algorithms for trees often generalize to algorithms for graphs of small treewidth



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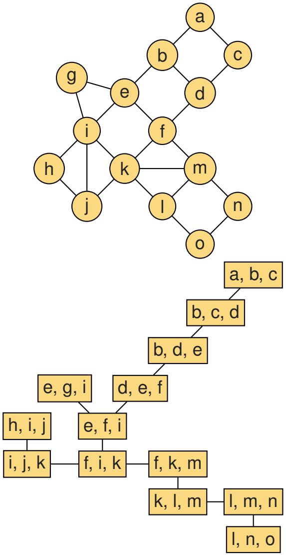

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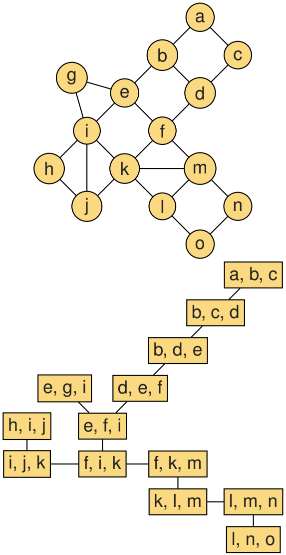

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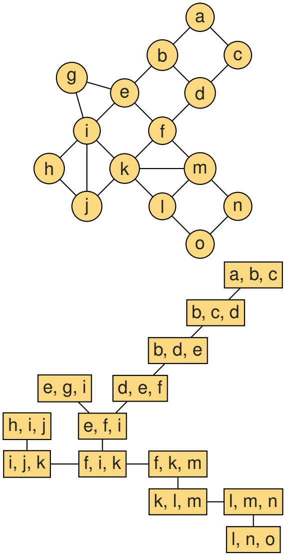

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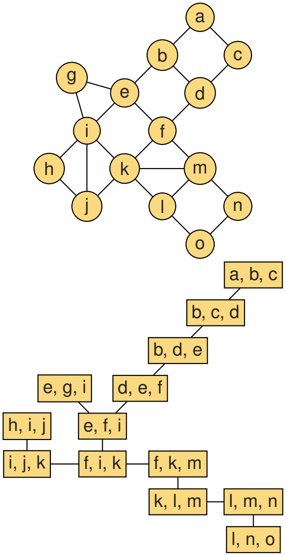
  - $2^{\mathcal{O}(k^3)} n$ time algorithm to compute an optimum-width tree decomposition [Bodlaender '96]
  - $2^{\mathcal{O}(k)} n$ time for 2-approximation [K. '21]
  - $n^{\mathcal{O}(1)}$ time for $\mathcal{O}(\sqrt{\log k})$-approximation [Feige,Hajiaghayi,Lee'08]



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**Question [Bodlaender '93, Dvořák, Kupec & Tůma '14, Alman, Mnich & Vassilevska Williams '20]**

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$2^{\mathcal{O}(k)} \log n$ amortized update time 9-approximate tree decomposition.

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- Arguably optimal update time: $\Omega(\log n)$ needed for dynamic forests [Patrascu&Demaine'04]

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There is data structure that:

- is initialized with an integer $k$ and an edgeless $n$-vertex graph $G$
- supports edge insertions/deletions in amortized time $2^{\mathcal{O}(k)} \log n$ under the promise that $\mathrm{tw}(G) \leq k$
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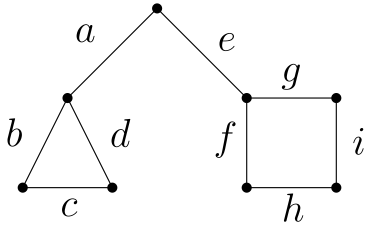
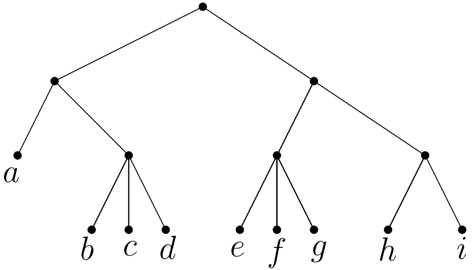
$\Rightarrow$ Dynamic Courcelle's theorem in $f(k) \cdot \log n$ amortized update time

# The algorithm

# Maintained decomposition




- Branch decomposition: Rooted tree whose leaves correspond to the edges of the graph

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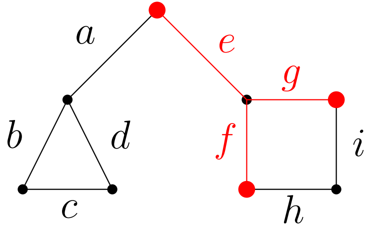


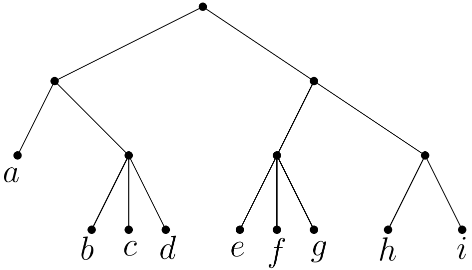


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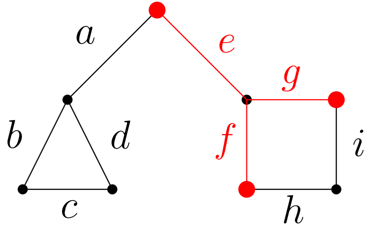


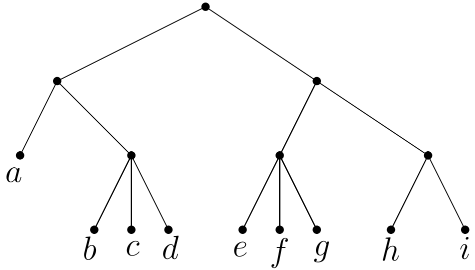


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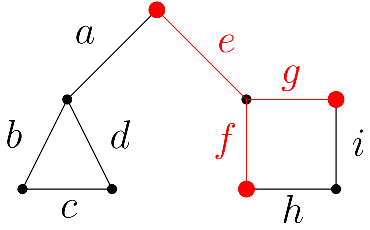


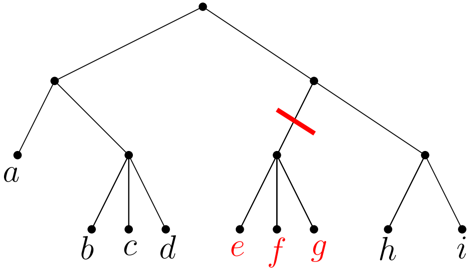


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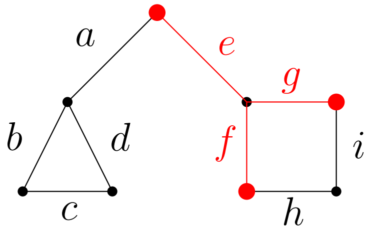


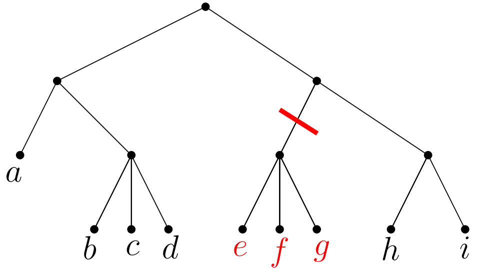


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  $\Rightarrow$ Boundaries have size $\mathcal{O}(k)$

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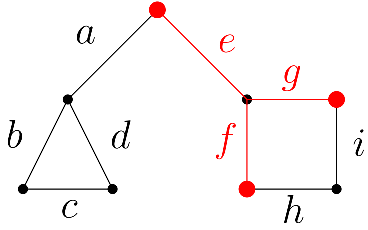


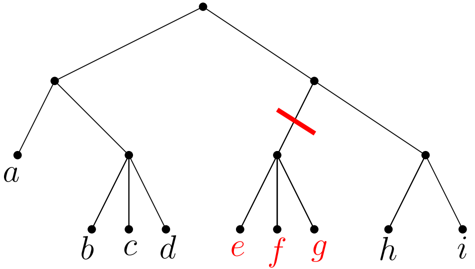


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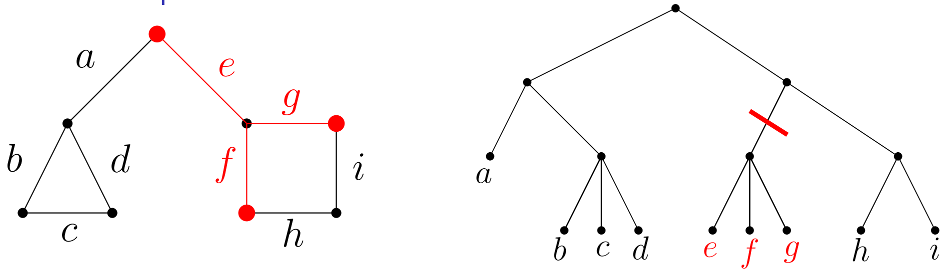


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  - $\Rightarrow$ Boundaries have size $\mathcal{O}(k)$
- Also: Degree at most $2^{\mathcal{O}(k)}$
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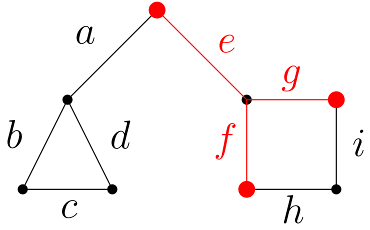


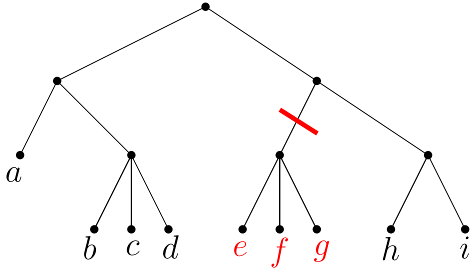


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- Depth at most $2^{\mathcal{O}(k)} \log n$

# Local rotations

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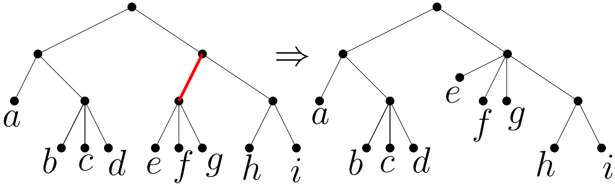
1. **Contraction**: Given an edge *uv* of the decomposition, contract it.



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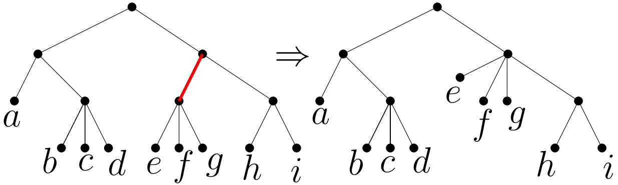
   - Maintains downwards well-linkedness



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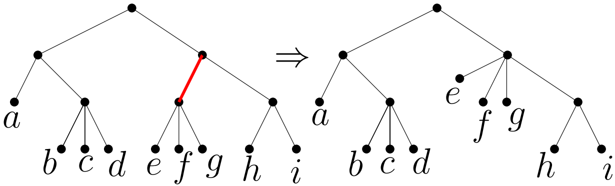
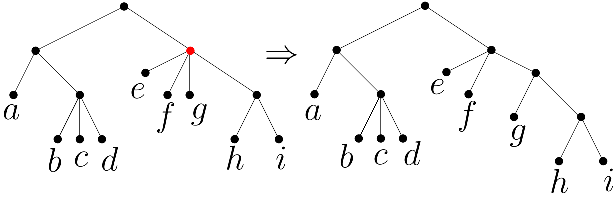



2. **Splitting**: Given a node of degree $> f(k) = 2^{\mathcal{O}(k)}$, locally split it to multiple nodes



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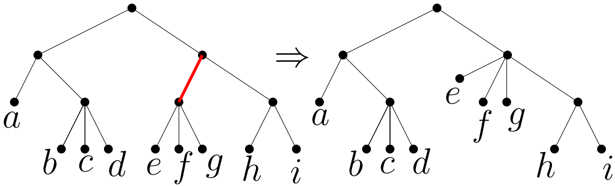


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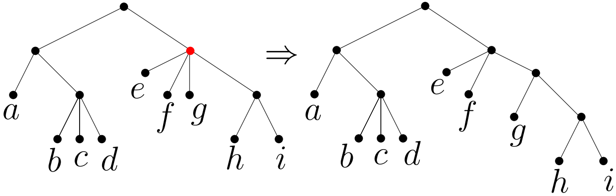
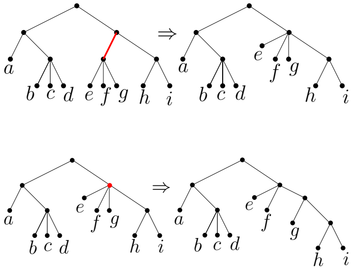
   - Lemma: Can be done so that downwards well-linkedness is maintained



# Approach

- **Idea**: Implement splay-tree-like rotations with the local rotations



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- **Idea**: Implement splay-tree-like rotations with the local rotations
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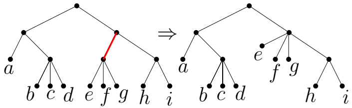


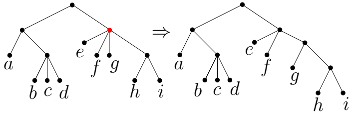

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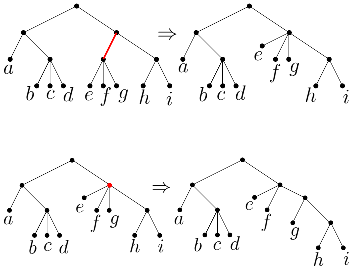

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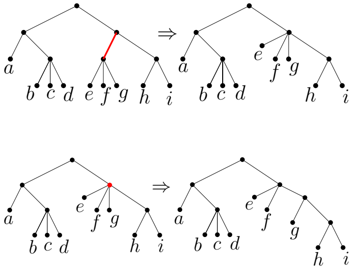
  1. rotate the self-loops *u* and *v* to be children of the root
  2. insert *uv* as another child of the root
  3. re-balance



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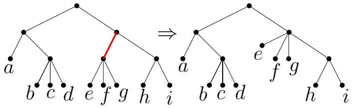


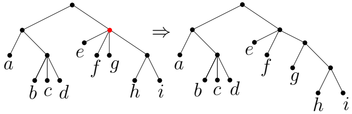

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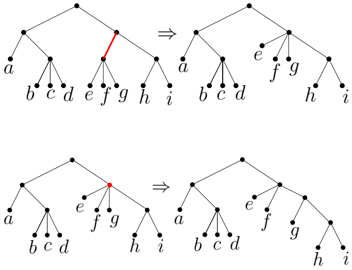
- **Post-processing:** Replace each node of size $2^{\mathcal{O}(k)}$ by a tree decomposition of width $\mathcal{O}(k)$



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Thank you!