Minor Containment and Disjoint Paths in almost-linear time

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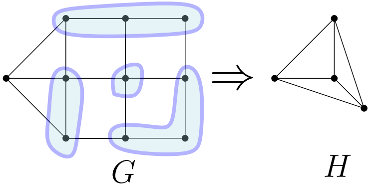
Minors of graphs

- A graph H is a minor of a graph G if H can be obtained from G by
  - Vertex deletions
  - Edge deletions
  - Edge contractions



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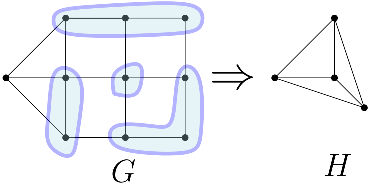
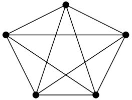
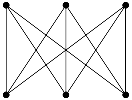


Theorem (Kuratowski-Wagner, 1930, 1937)

A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.

Planar $\Leftrightarrow$ no  or 

The Graph Minors Series

Theorem (Robertson & Seymour, 1984-2004)

Let $\mathcal{C}$ be a minor-closed graph class. There exists a finite set of graphs $\mathcal{H}$, s.t. a graph G is in $\mathcal{C}$ if and only if G does not contain a graph from $\mathcal{H}$ as a minor.

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- More generally, an $f(H) \cdot n^3$ time algorithm for Rooted Minor Containment
  - $\Rightarrow$ $f(k) \cdot n^3$ time algorithm for the k-Disjoint Paths problem

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Our Algorithm

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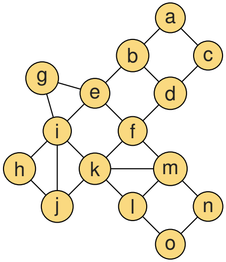
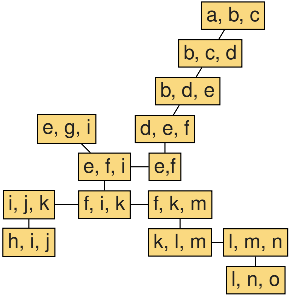
- Treewidth of a graph: Parameter between 0 and $n-1$ measuring how tree-like the graph is



Graph G
Treewidth 2



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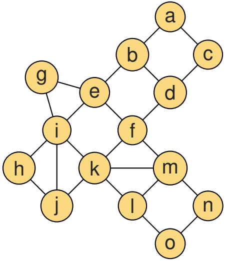

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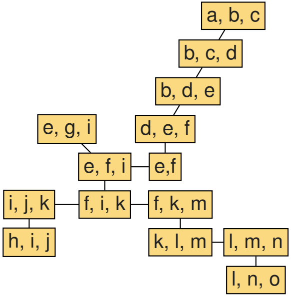

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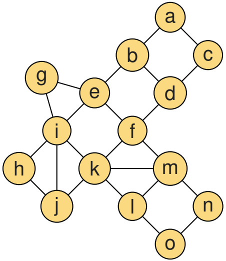

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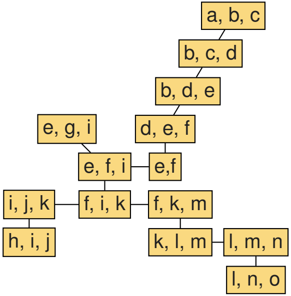

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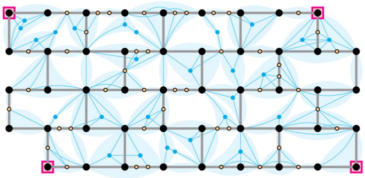

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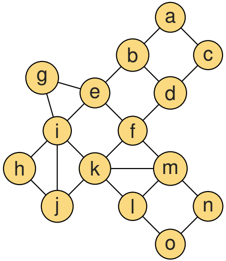

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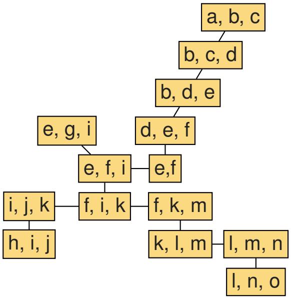

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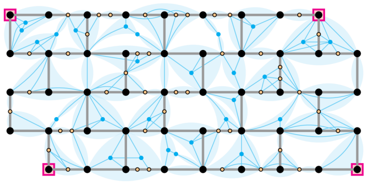

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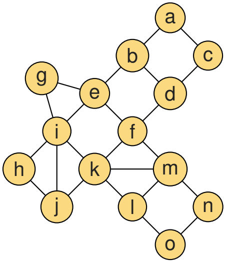

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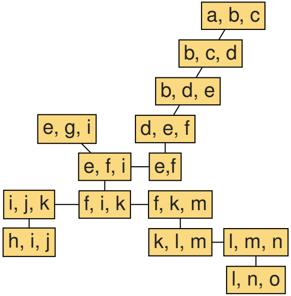

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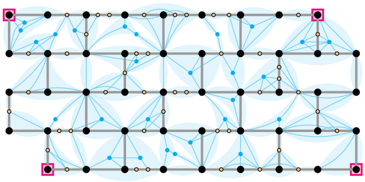

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