A Single-Exponential Time 2-Approximation Algorithm for Treewidth

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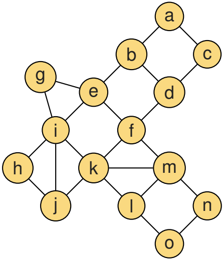
Treewidth

- Measures how close a graph is to a tree



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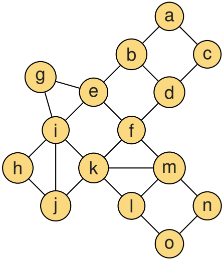

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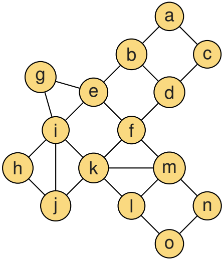

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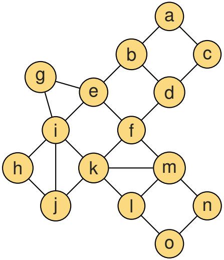

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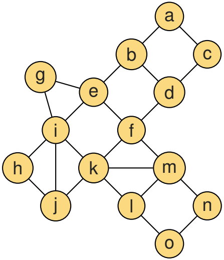

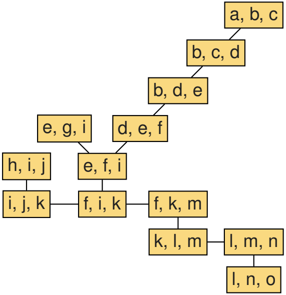

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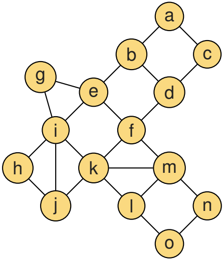

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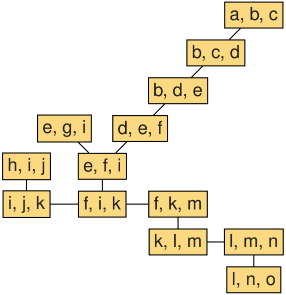

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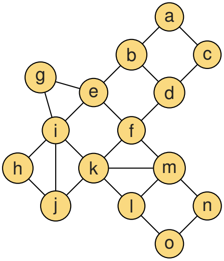

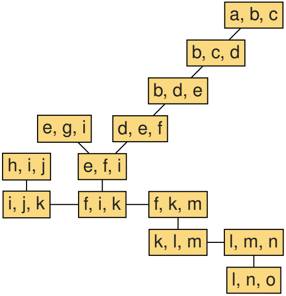

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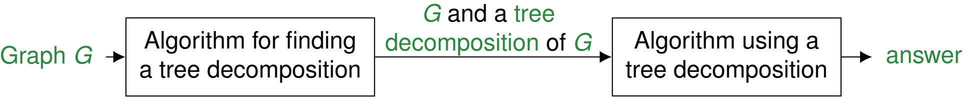

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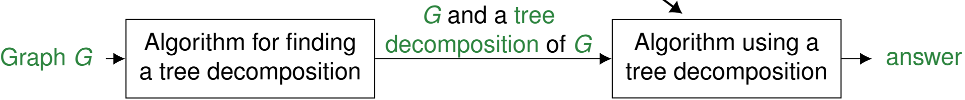

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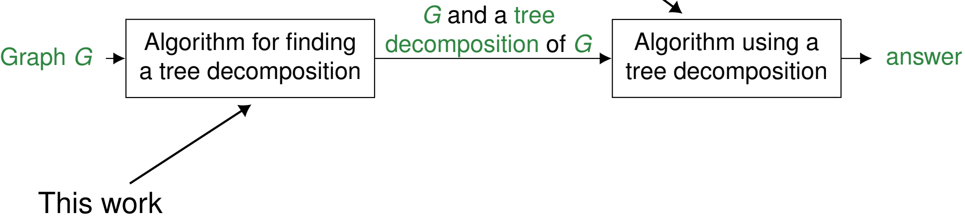

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New approach for approximating treewidth

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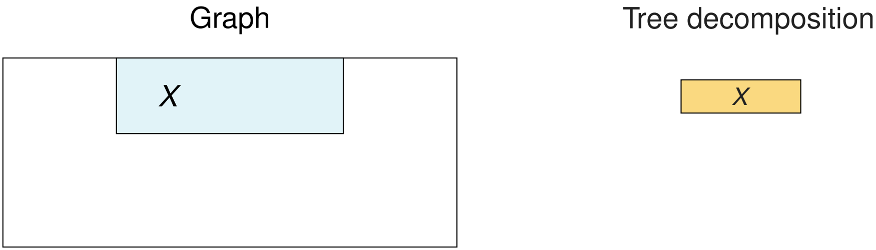

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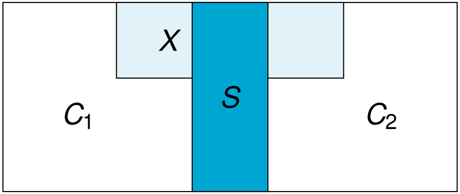
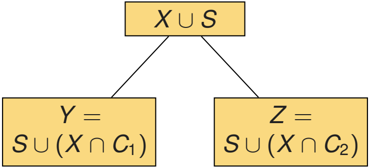


Separator S with components C_1 and C_2

Tree decomposition



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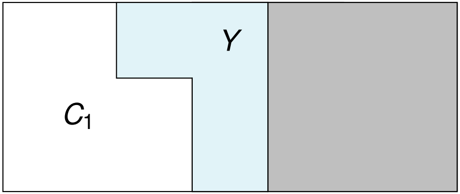

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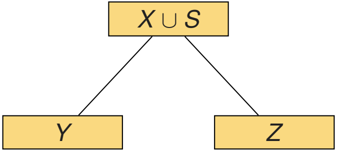

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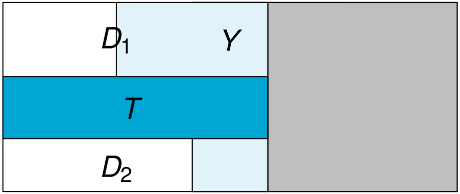

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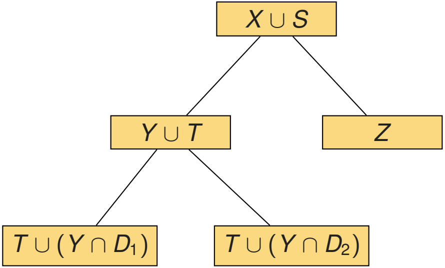

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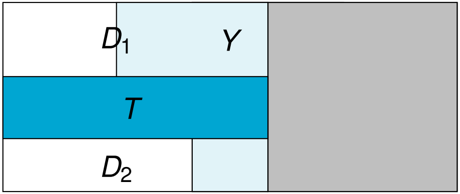

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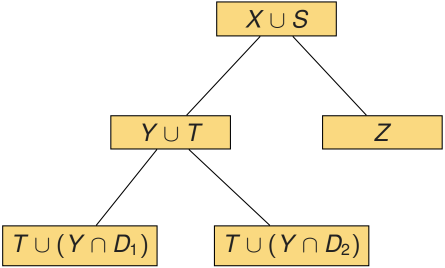

- Barrier at approximation ratio 3

The algorithm of this work

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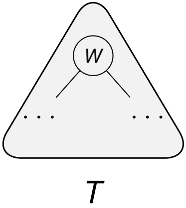
   - Efficient implementation by amortizatized analysis of the improvements and dynamic programming over the tree decomposition

The improvement operation

- Let W be the largest bag



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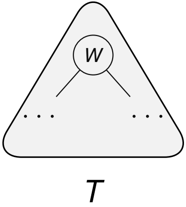

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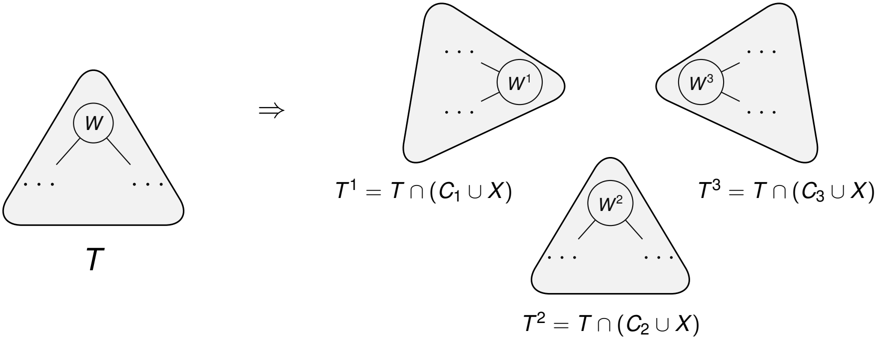

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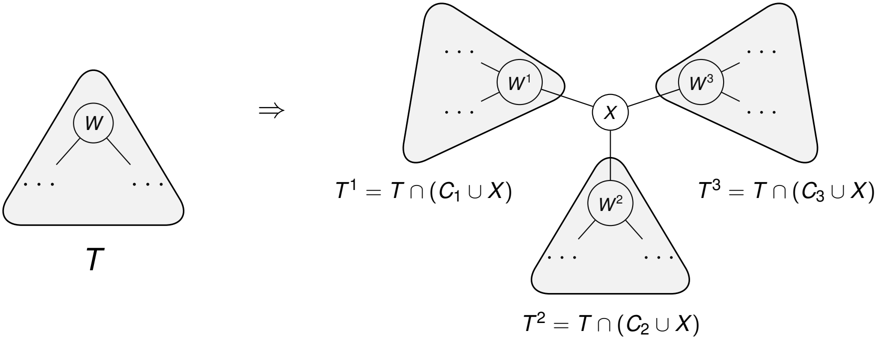

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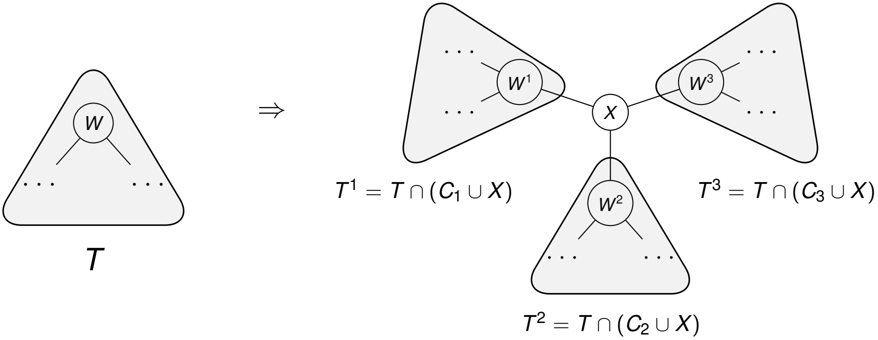

Except that vertices in X may violate the connectedness condition

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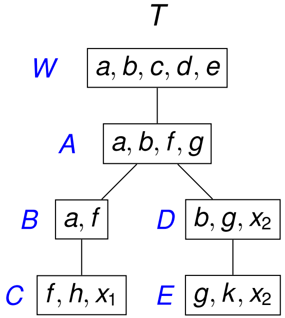

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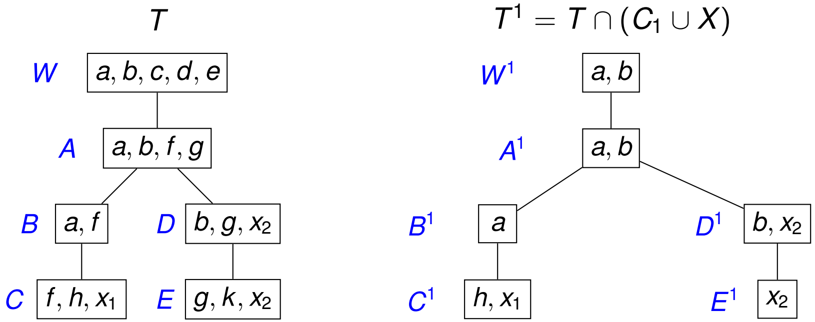

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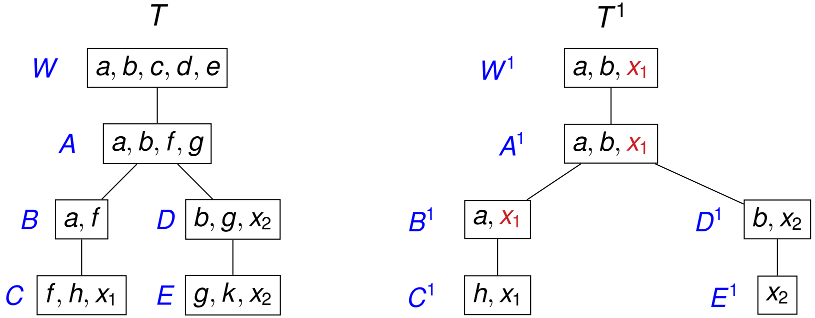

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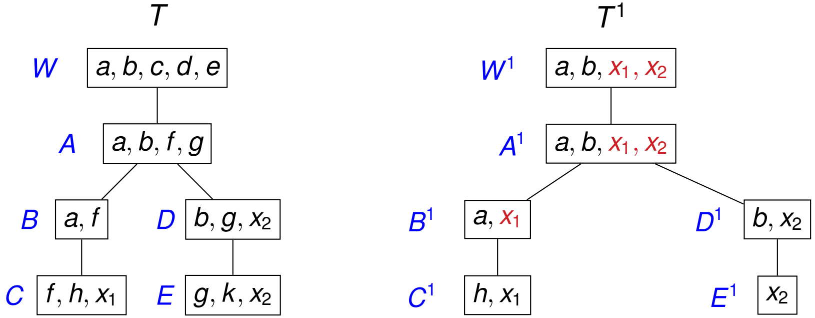

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- For the bag W, $|W^i| < |W|$ is ensured by the definition of the balanced separator

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If the balanced separator X is chosen according to specific criteria, then $|B^i| \leq |B|$ for all bags B and each i.

- $|B^i| = |B|$ holds only in a degenerate case where we can throw B^j for $j \neq i$ away
- For the bag W, $|W^i| < |W|$ is ensured by the definition of the balanced separator

$\Rightarrow$ The number of bags of size $|W|$ decreases

Conclusion

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- Subsequently, we extended the approach to also branchwidth of symmetric submodular functions [Fomin and K., STOC'22]
- Open problem: Is there a $2^{\mathcal{O}(k^c)} n^{\mathcal{O}(1)}$ time exact algorithm for treewidth, where $c < 3$? (or even a better than 2-approximation?)

The end

Thank you for your attention!

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