Dynamic Treewidth

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UNIVERSITY OF BERGEN

based on joint work with Konrad Majewski, Wojciech Nadara, Michał Pilipczuk, and Marek Sokołowski, University of Warsaw

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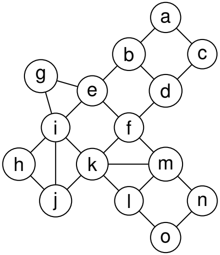
4. [Henzinger&King'99]: $\mathcal{O}(\log^3 n)$ amortized time

Treewidth



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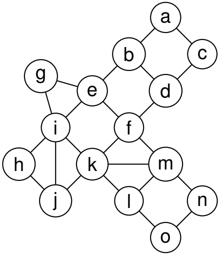

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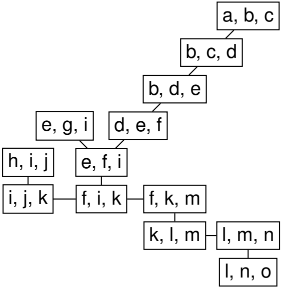

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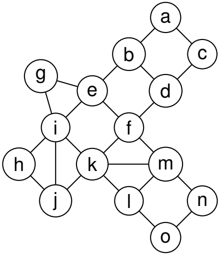

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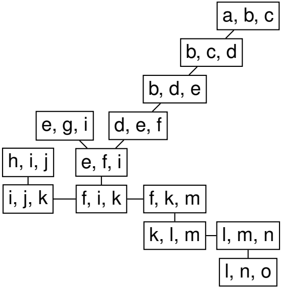

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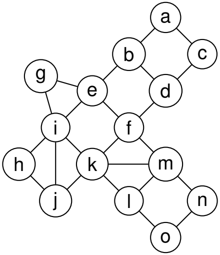

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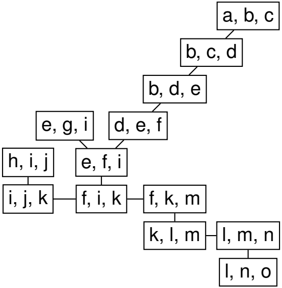

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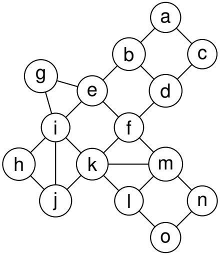

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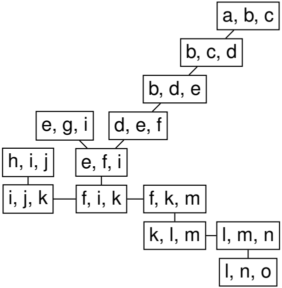

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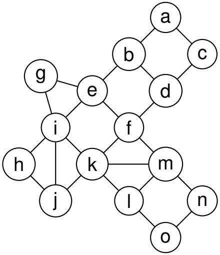

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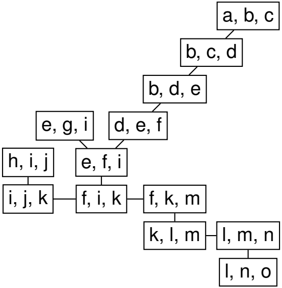

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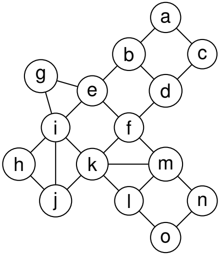

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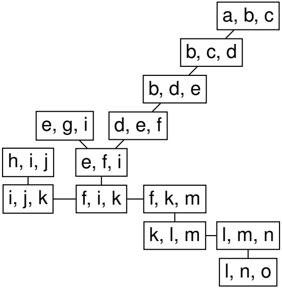

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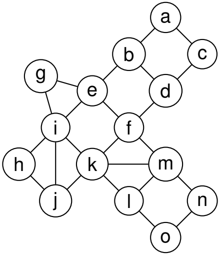

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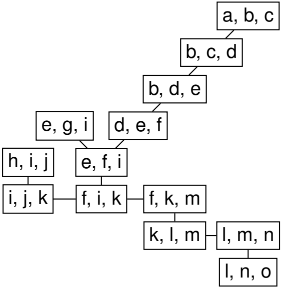

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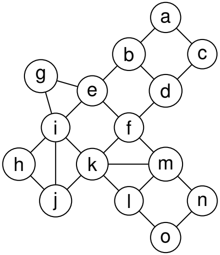

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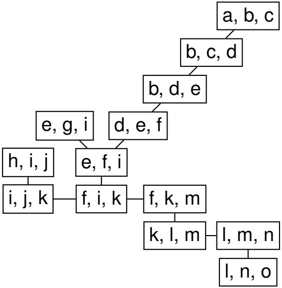

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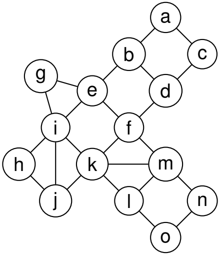

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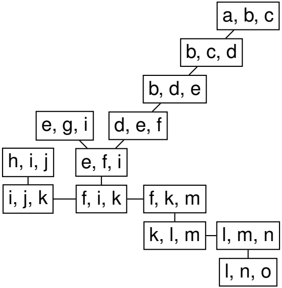

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[Robertson & Seymour'84, Bertele & Brioschi'72, Halin'76]

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- [Goranci,Saranurak,Tan’21]: $n^{o(1)}$ amortized time $n^{o(1)}$-approximate tree decomposition. Not suitable for dynamic programming.

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There is a data structure that is initialized with an integer k and an empty n-vertex graph G, and maintains a tree decomposition of G of width at most $6k + 5$ under edge additions and deletions in amortized update time $\mathcal{O}_k(2^{\sqrt{\log n} \log \log n})$, under the promise that the treewidth of G never exceeds k.

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- the data structure can maintain the run of any tree automaton with evaluation time $\mathcal{O}_k(1)$ within the same running time
- the data structure persists even when the treewidth of G exceeds k, in that case returning a marker “Treewidth too large” instead of maintaining the automaton

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- Use dynamic treewidth data structure with this k and a tree automaton that tests for H as a minor by dynamic programming □

The algorithm

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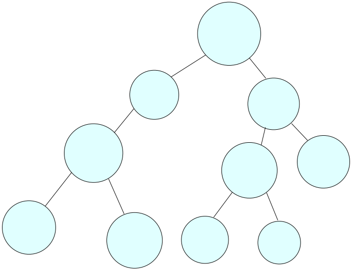

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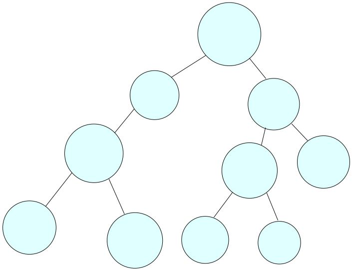

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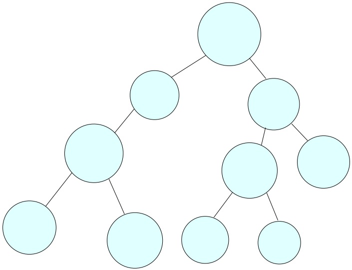

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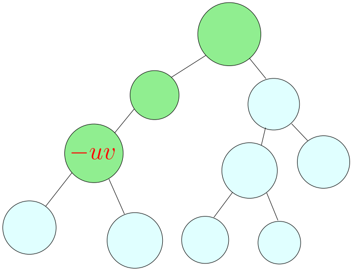

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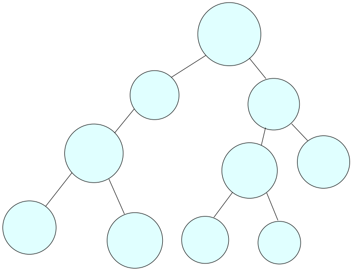

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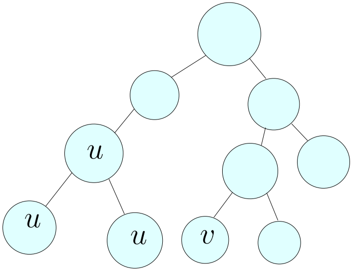

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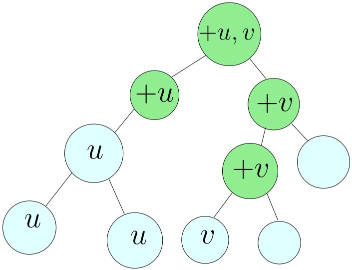

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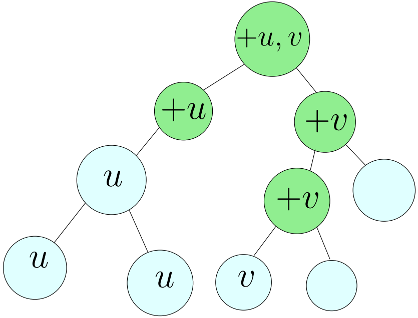

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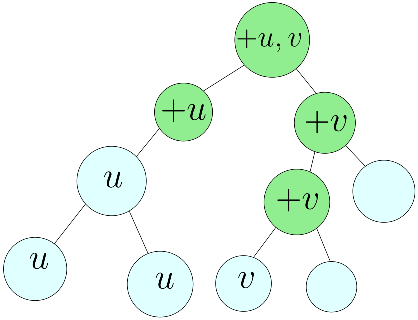
- The width can become more than $6k + 5$ on the green bags!



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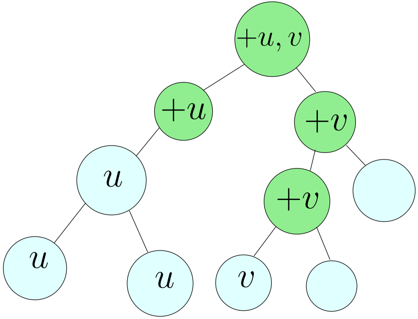
- The width can become more than $6k + 5$ on the green bags!
- Solution: a *Refinement operation* to re-compute the tree decomposition on these bags



Refinement operation

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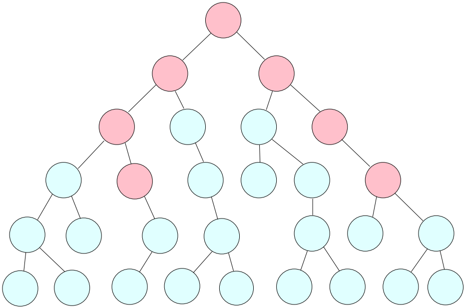
- Refinement operation is given a *prefix* T_{pref} of the tree decomposition that contains all bags of width $> 6k + 5$



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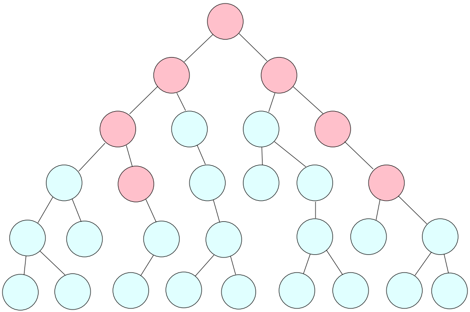

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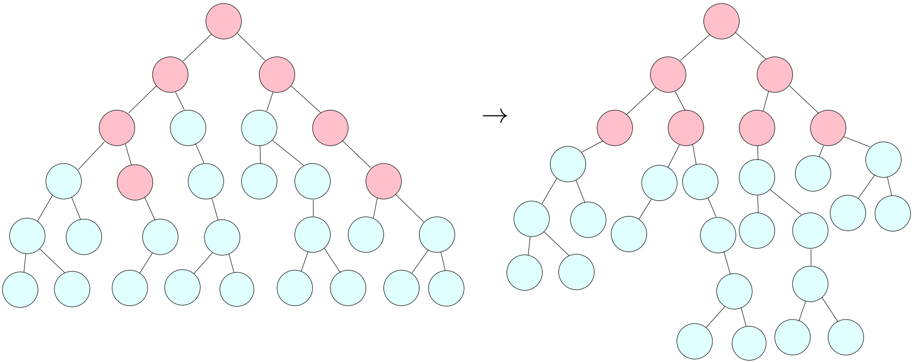

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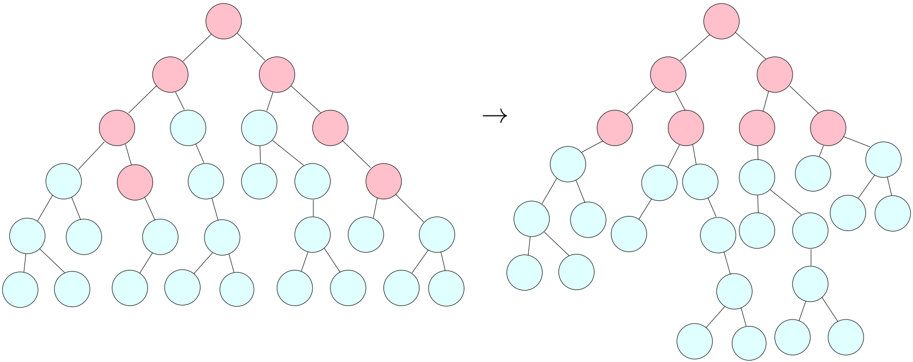

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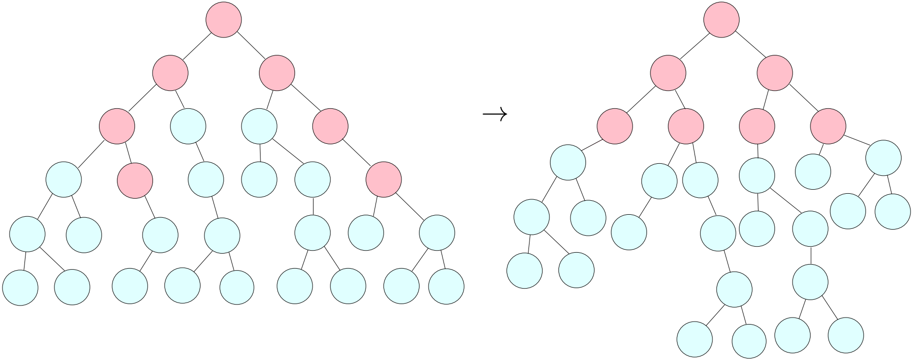
- Builds on the improvement operation of [K & Lokshtanov'23], also uses the dealternation lemma of [Bojańczyk&Pilipczuk'22] and Bodlaender-Hagerup-lemma



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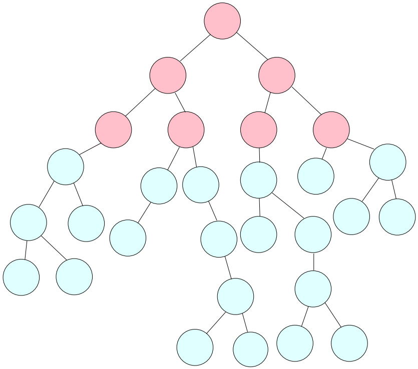

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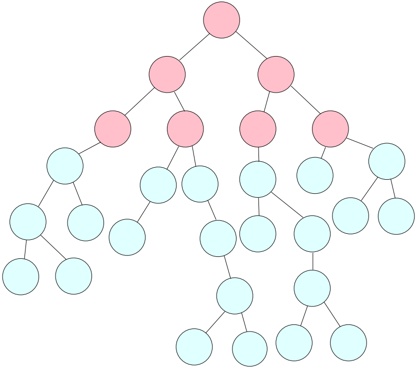
- Refinement operation can increase the depth by $\mathcal{O}(\log n)$



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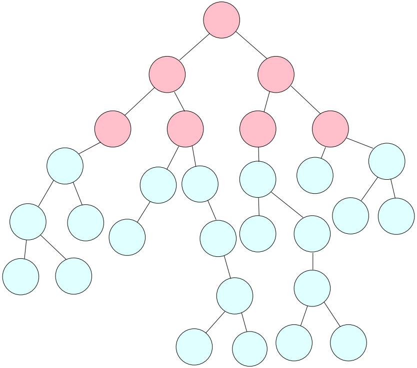

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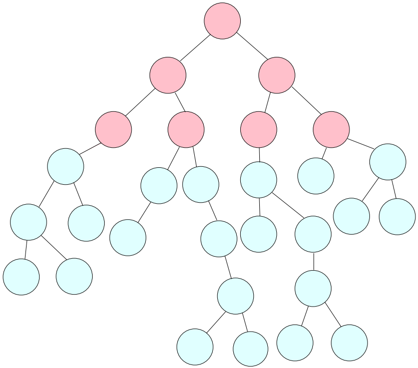
- Solution: A depth-reduction scheme by using the refinement operation and a potential function



Potential function

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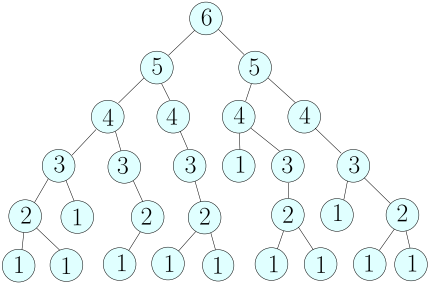
- Potential function of form $\phi(T) = \sum_{t \in V(T)} k^{10 \cdot |B_t|} \cdot \text{height}(t)$



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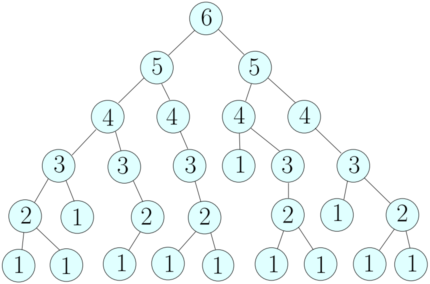

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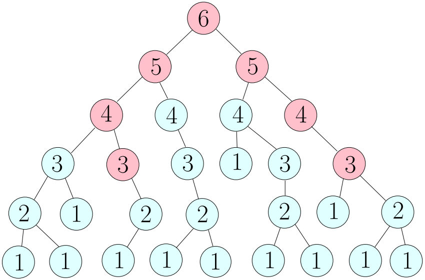

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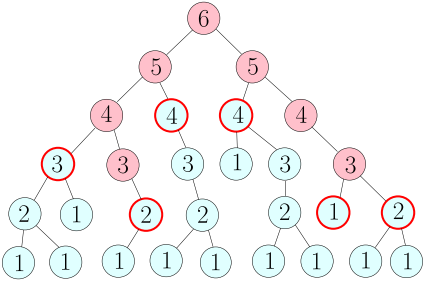

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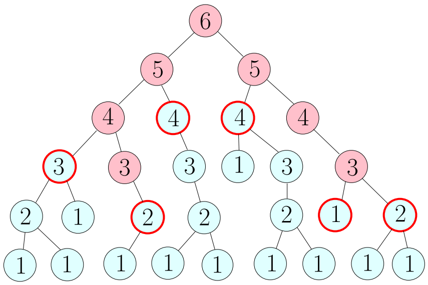

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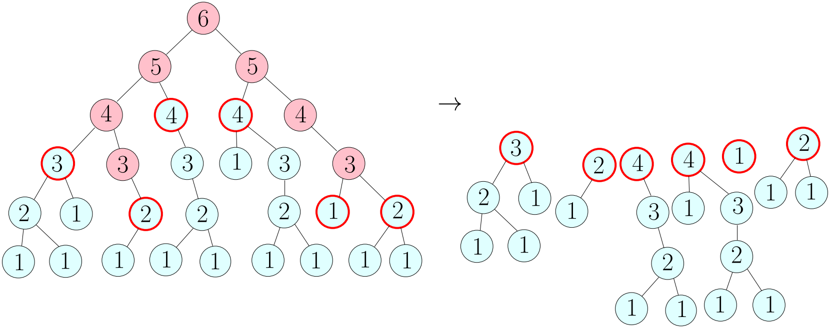

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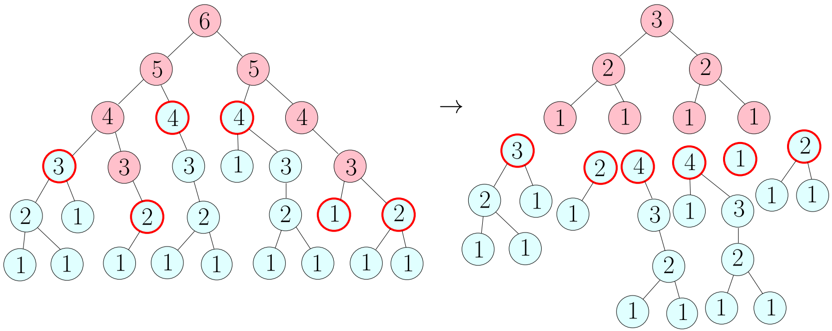

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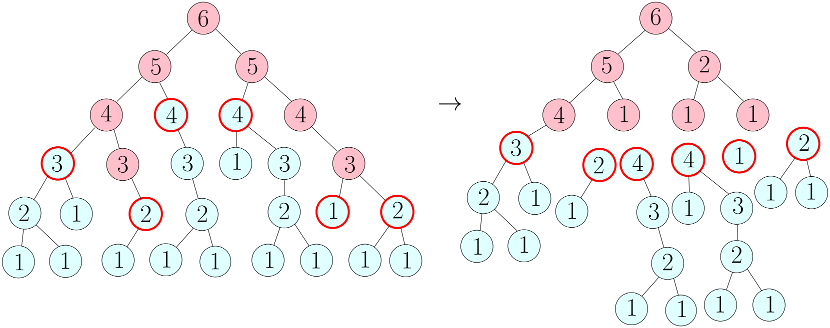

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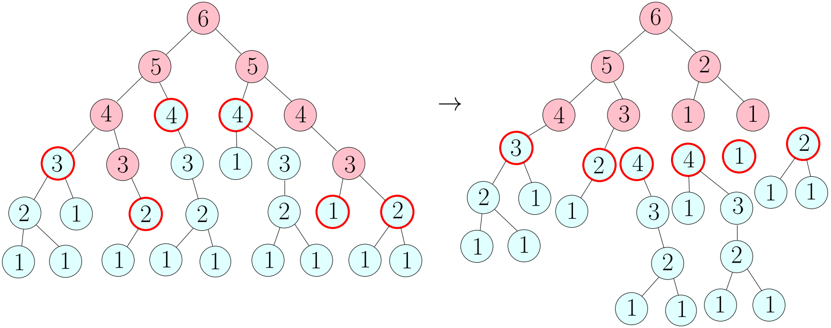

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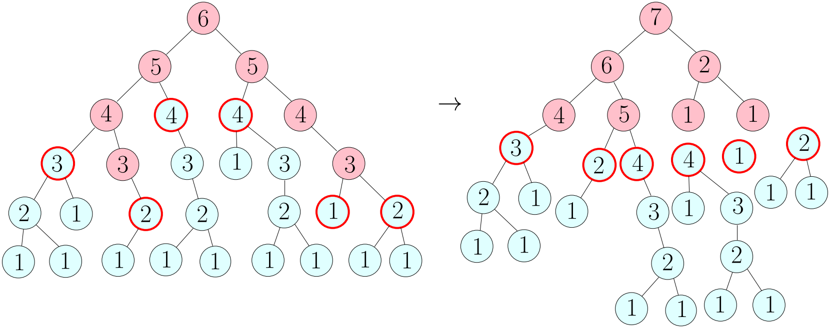

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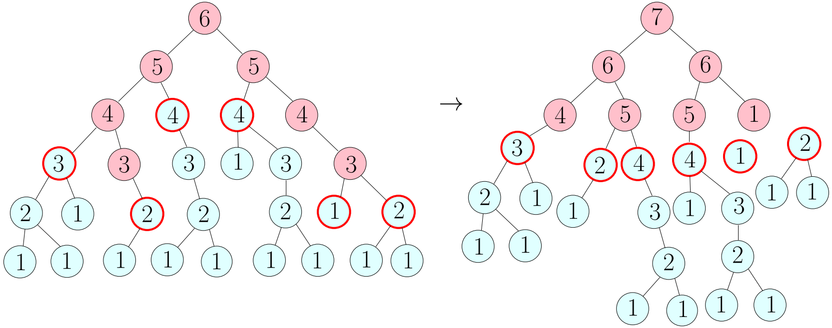

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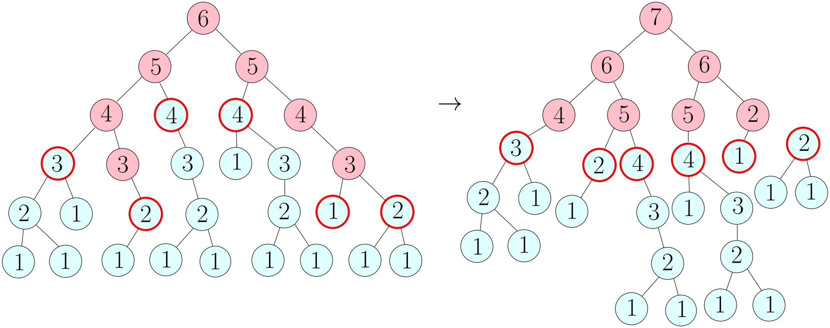

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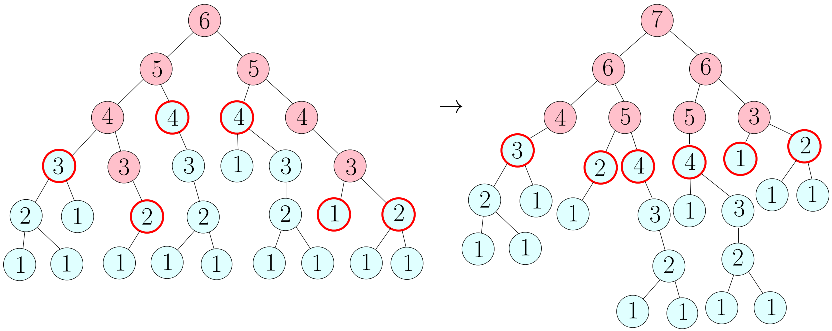

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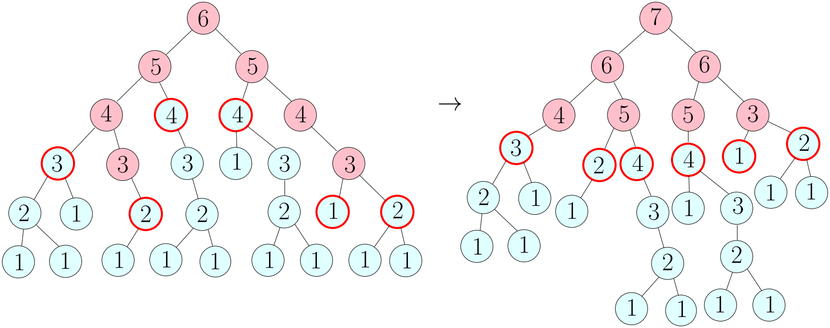

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$\Rightarrow$ Can keep depth at most $2^{\mathcal{O}_k(\sqrt{\log n \log \log n})}$ with amortized time complexity $2^{\mathcal{O}_k(\sqrt{\log n \log \log n})}$

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Thank you!