Linear-Time Algorithms for k-Edge-Connected Components, k-Lean Tree Decompositions, and More

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BWAG ’25

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- $\mathcal{O}(k^2 m \log m)$ [Gabow '91], $\mathcal{O}(m \operatorname{polylog} m)$ [Karger '96]

k-Lean Tree Decompositions and More

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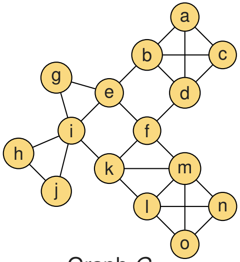
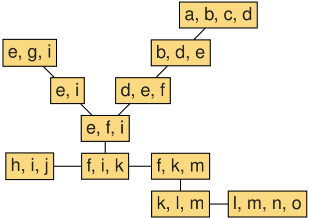
  - Previously $k^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ [Cygan, Komosa, Lokshtanov, Pilipczuk, Pilipczuk, Saurabh, Wahlström '21]
  - and $k^{\mathcal{O}(k)} m^{1+o(1)}$ [Anand, Lee, Li, Long, Saranurak '24] (suboptimal unbreakability parameters)

k-Lean Tree Decompositions



Graph G



A 3-lean tree decomposition of G

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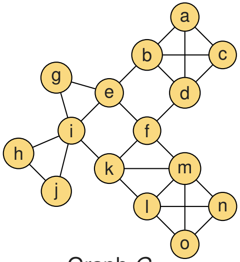

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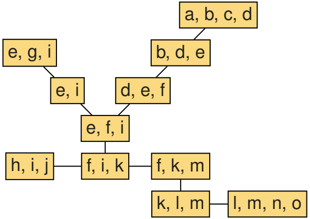

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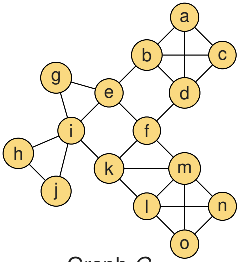

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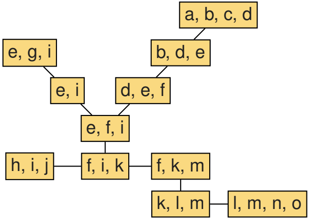

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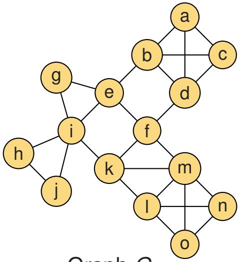

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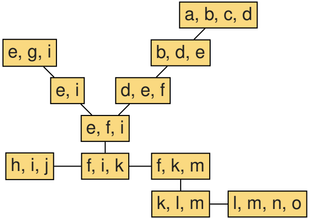

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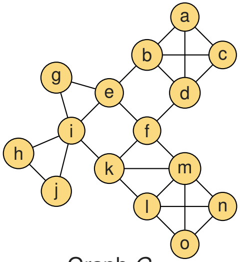

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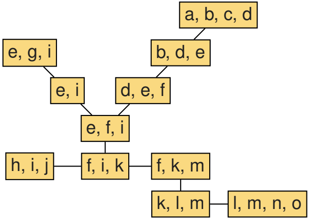

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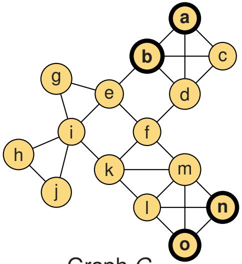

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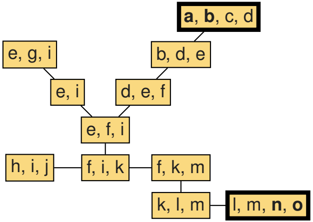

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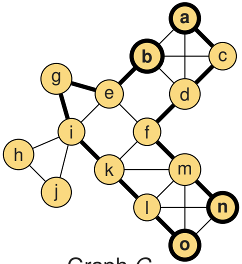

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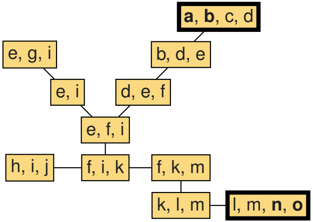

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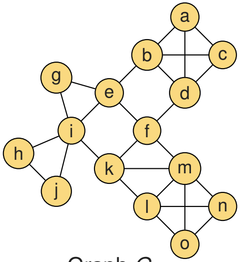

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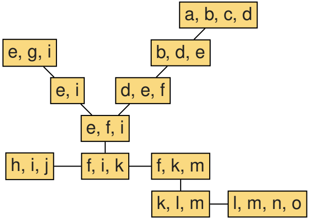

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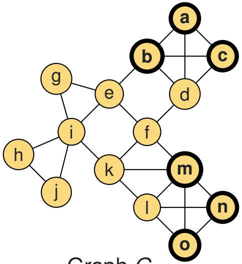

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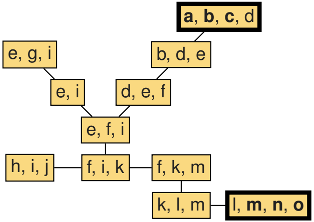

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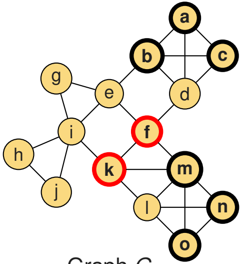

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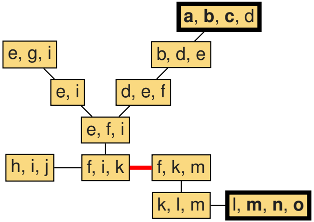

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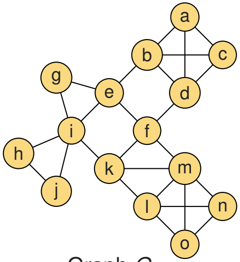

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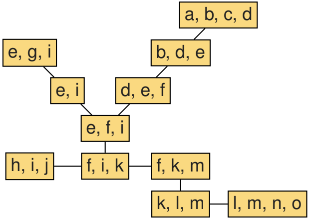
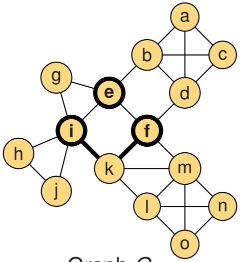


A 3-lean tree decomposition of G

- Tree decomposition:
  1. All vertices and edges are covered by bags
  2. For each vertex v, the bags containing v form a connected subtree
- k-lean:
  1. The adhesions (i.e. intersections of adjacent bags) have size $< k$
  2. For all pairs of bags X_1, X_2 and subsets $Y_1 \subseteq X_1$, $Y_2 \subseteq X_2$ with $|Y_1| = |Y_2| \leq k$, the sets Y_1 and Y_2 can be linked by vertex-disjoint paths if and only if there is no (X_1, X_2)-adhesion of size $< |Y_1| = |Y_2|$

k-Lean Tree Decompositions



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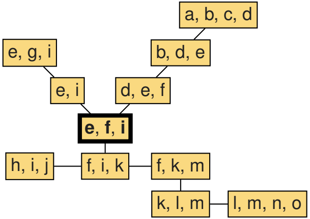

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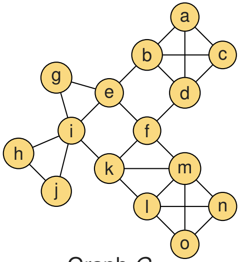

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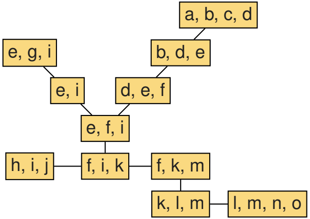

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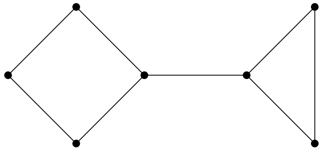
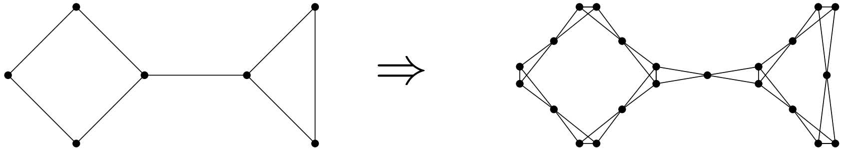
- Defined by [Thomas '90] (for $k = \infty$), and [Carmesin, Diestel, Hamann, and Hundertmark '14]

Reducing k-edge-connected components to k-lean tree decomposition



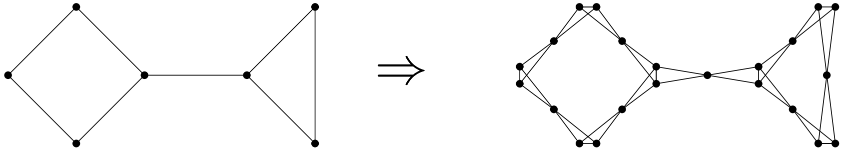
Reducing k-edge-connected components to k-lean tree decomposition

- Replace vertices by cliques of size k
- Create vertex for each edge and connect to the cliques corresponding to its endpoints



Reducing k-edge-connected components to k-lean tree decomposition

- Replace vertices by cliques of size k
- Create vertex for each edge and connect to the cliques corresponding to its endpoints
- Resulting k-lean tree decomposition gives a k-Gomory-Hu tree



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Part 2: The improver algorithm

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Lemma

If there is improver algorithm with running time $f(k) \cdot m$, then there is an algorithm that in time $k^{\mathcal{O}(1)} \cdot f(k) \cdot m$ computes a k-lean tree decomposition.

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Case 2: No matching of size $\Omega(n/\text{poly}(k)) \Rightarrow$ similar recursion in some other way...

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There is an improver algorithm with running time $k^{\mathcal{O}(k^2)} m$.

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There is an improver algorithm with running time $k^{\mathcal{O}(k^2)} m$.

Idea:

- Slowly improve the properties of the input decomposition (in 6 consecutive steps)

The improver algorithm

Goal

Want to compute a superbranch decomposition T of G so that

- T has adhesion size $< k$,
- torsos of T are $(2^{\mathcal{O}(k)}, k)$-unbreakable, and
- internal separations of T are doubly well-linked

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Lemma

A k-lean tree decomposition of an (s, k)-unbreakable graph can be computed in $s^{\mathcal{O}(k)} m$ time.

Focus

$2^{\mathcal{O}(k^2)}m$-time algorithm for a problem with:

Input: Graph G and superbranch decomposition T of G s.t.

- T has adhesion size $< 2k$, and
- torsos of T are $(2k, k)$-unbreakable in G.

Output: Superbranch decomposition T of G s.t.

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Step 1: Downwards well-linkedness

Input: Graph G and superbranch decomposition T of G s.t.

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Step 2: Upwards k-well-linkedness

Input: Graph G and a superbranch decomposition T of G s.t.

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Output: Superbranch decomposition T of G s.t.

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Step 3: Tangle k-unbreakability

Input: Graph G and a superbranch decomposition T of G s.t.

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- T has adhesion size $< 2k$,
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Step 4: Small adhesions

Input: Graph G and a superbranch decomposition T of G s.t.

- T has adhesion size $< 2k$,
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Output: Superbranch decomposition T of G s.t.

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Step 5: Unbreakable torsos

Input: Graph G and a superbranch decomposition T of G s.t.

- T has adhesion size $< k$,
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