Dynamic Treewidth in Logarithmic Time

Tuukka Korhonen

ARCO

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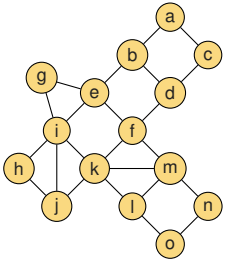
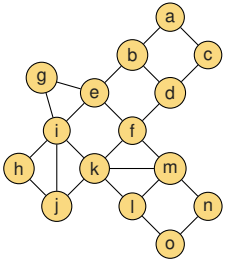
Treewidth



Graph G

Treewidth



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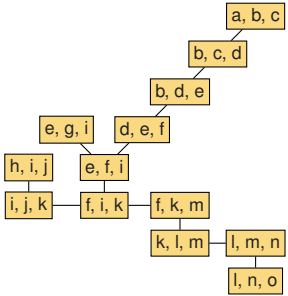
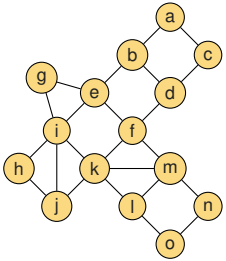


A tree decomposition of G

Treewidth



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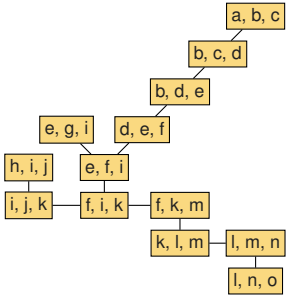

A tree decomposition of G

1. Every vertex should be in a bag

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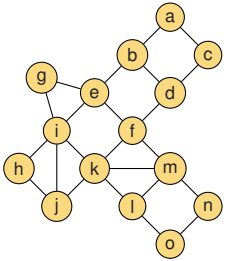

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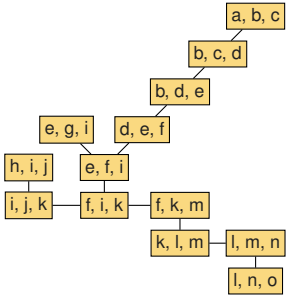

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2. Every edge should be in a bag

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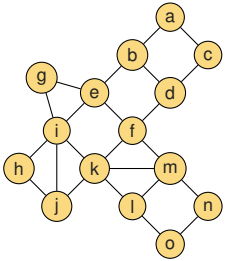

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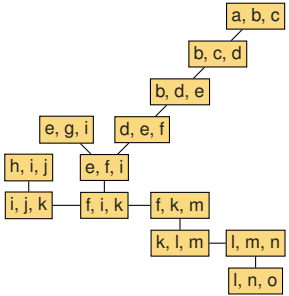

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3. For every vertex v, the bags containing v should form a connected subtree

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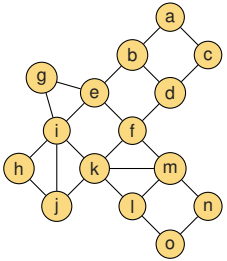

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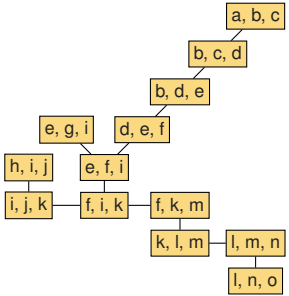

A tree decomposition of G
Width = 2

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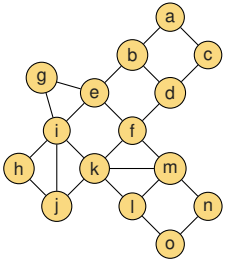

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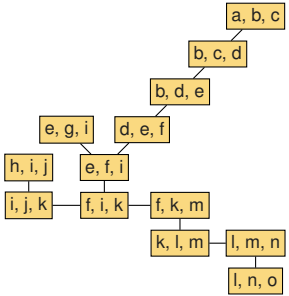

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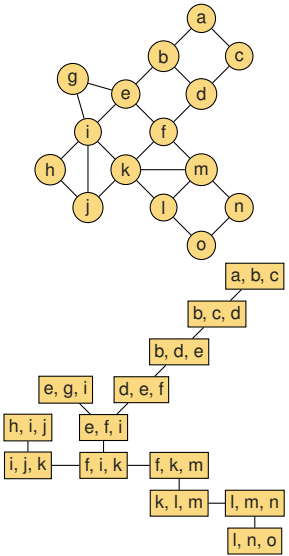
4. Width = maximum bag size -1
5. Treewidth of G = minimum width of tree decomposition of G

[Robertson & Seymour'84, Arnborg & Proskurowski'89, Bertele & Brioschi'72, Halin'76]

Why treewidth?

- Algorithms for trees often generalize to algorithms for graphs of small treewidth



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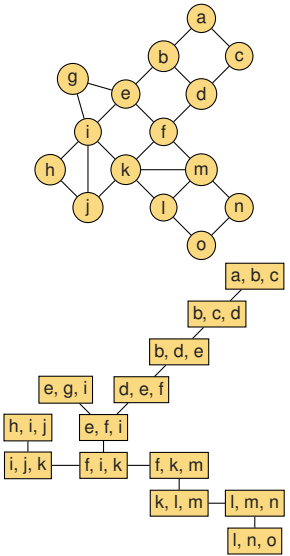

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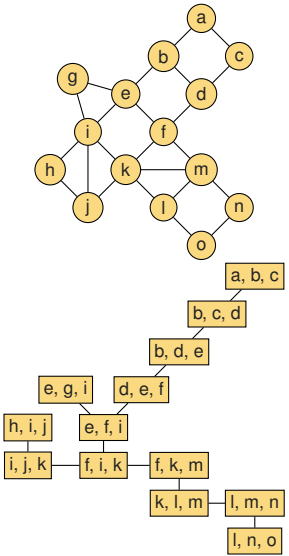

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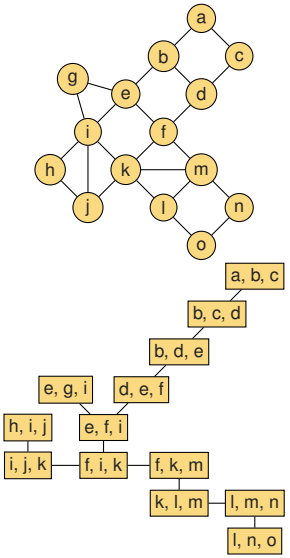

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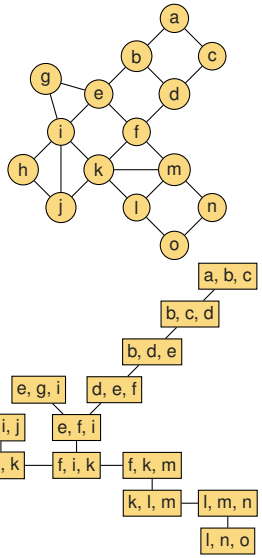
  - $2^{\mathcal{O}(k^3)}n$ time algorithm to compute an optimum-width tree decomposition [Bodlaender '96]
  - $2^{\mathcal{O}(k)}n$ time for 2-approximation [K. '21]
  - $n^{\mathcal{O}(1)}$ time for $\mathcal{O}(\sqrt{\log k})$-approximation [Feige,Hajiaghayi,Lee'08]



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- Arguably optimal update time: $\Omega(\log n)$ needed for dynamic forests [Patrascu&Demaine'04]

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There is data structure that:

- is initialized with an integer k and an edgeless n-vertex graph G
- supports edge insertions/deletions in amortized time $2^{\mathcal{O}(k)} \log n$ under the promise that $\mathrm{tw}(G) \leq k$
- maintains a tree decomposition of G of width at most $9 \cdot \mathrm{tw}(G) + 8$

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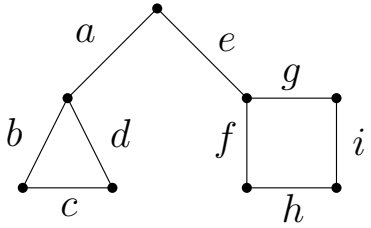
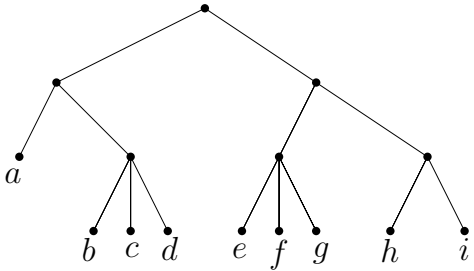
$\Rightarrow$ Dynamic Courcelle's theorem in $f(k) \cdot \log n$ amortized update time

The algorithm

Maintained decomposition





- Branch decomposition: Rooted tree whose leaves correspond to the edges of the graph

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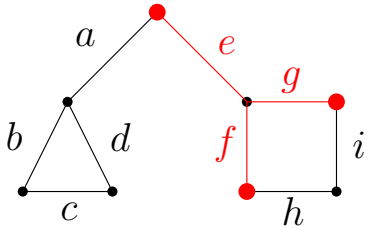

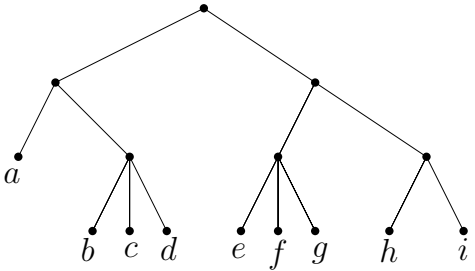

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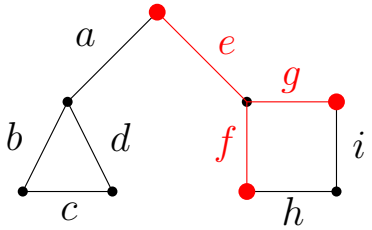

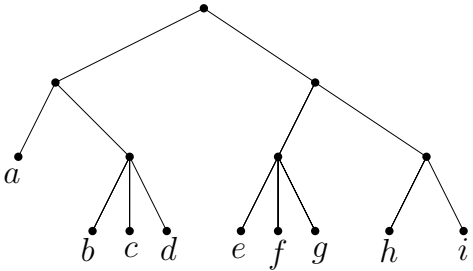

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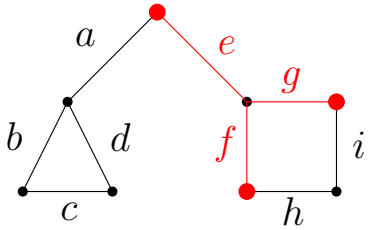

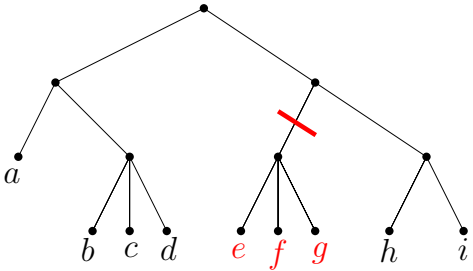

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- Want to maintain: Every edge set corresponding to a subtree is well-linked "downwards well-linkedness"

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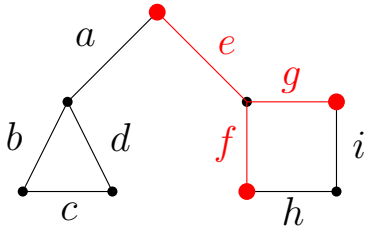

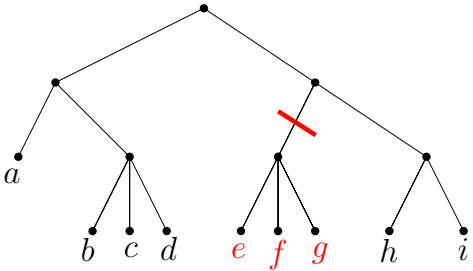

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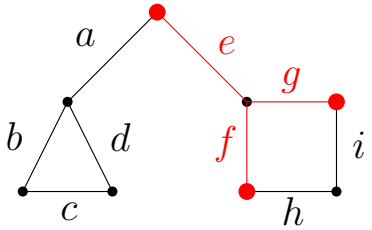

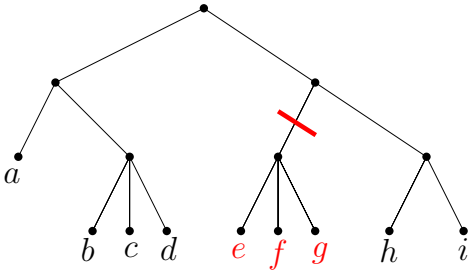

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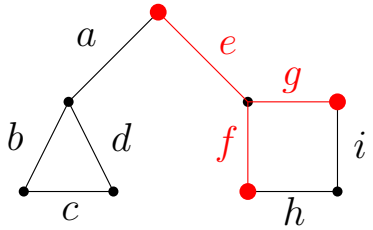
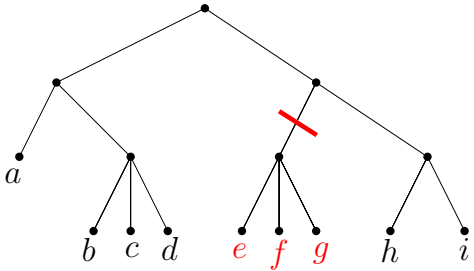
- Also: Degree at most $2^{\mathcal{O}(k)}$

Maintained decomposition





- Branch decomposition: Rooted tree whose leaves correspond to the edges of the graph
- Boundary $\partial(F)$ of a set of edges $F \subseteq E$: The vertices incident to edges from both F and $E \setminus F$.
- A set of edges $F \subseteq E$ is well-linked if it cannot be partitioned to (C_1, C_2) so that $|\partial(C_1)| < |\partial(F)|$ and $|\partial(C_2)| < |\partial(F)|$
- Want to maintain: Every edge set corresponding to a subtree is well-linked "downwards well-linkedness"
  $\Rightarrow$ Boundaries have size $\mathcal{O}(k)$
- Also: Degree at most $2^{\mathcal{O}(k)}$
  $\Rightarrow$ Tree decomposition of width $2^{\mathcal{O}(k)}$ (later optimize to $\mathcal{O}(k)$)

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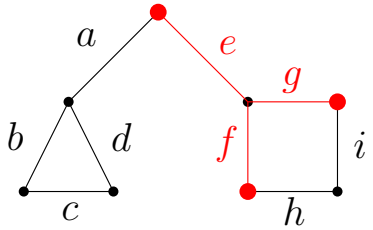

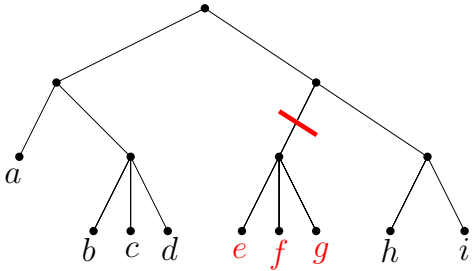

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- Depth at most $2^{\mathcal{O}(k)} \log n$

Local rotations

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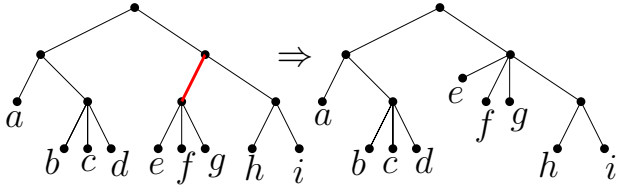

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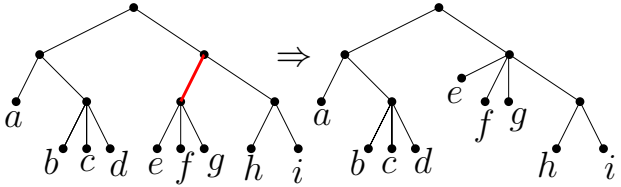

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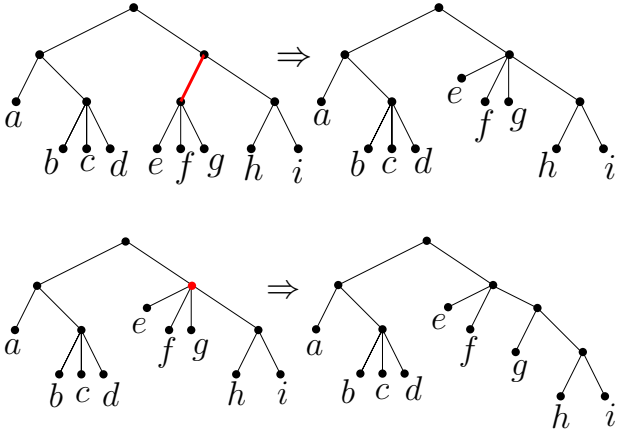

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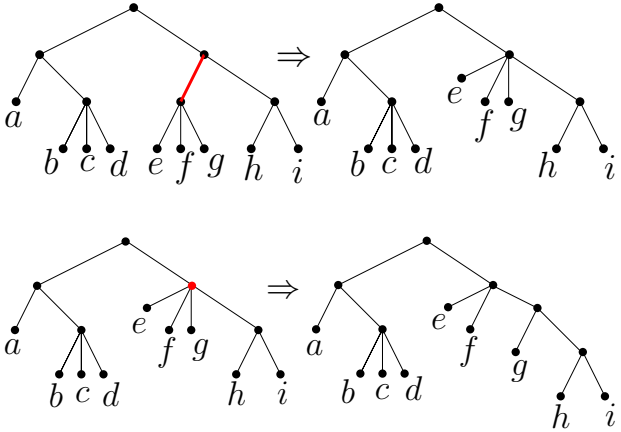
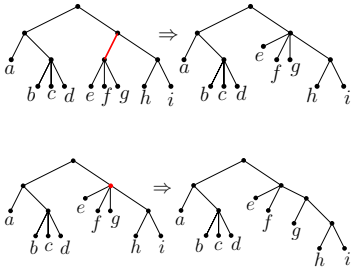
   - Lemma: Can be done so that downwards well-linkedness is maintained



Approach

- **Idea**: Implement splay-tree-like rotations with the local rotations



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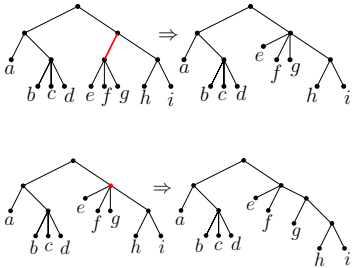

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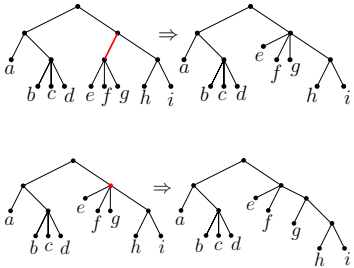

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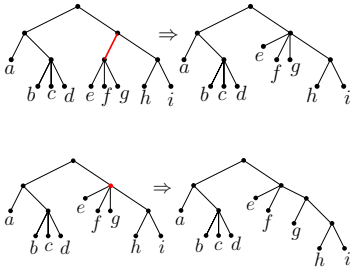

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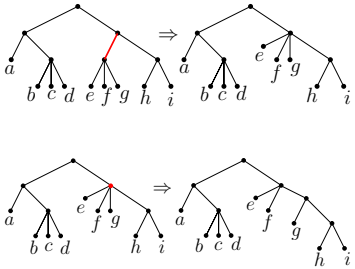

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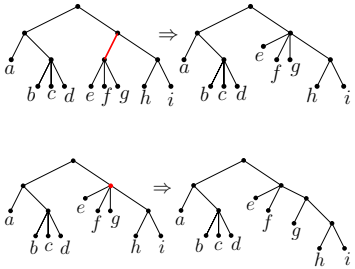
- **Post-processing:** Replace each node of size $2^{\mathcal{O}(k)}$ by a tree decomposition of width $\mathcal{O}(k)$



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Thank you!