Linear-Time Algorithms for k-Edge-Connected Components, k-Lean Tree Decompositions, and More

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- $\mathcal{O}(k^2 m \log m)$ [Gabow '91], $\mathcal{O}(m \operatorname{polylog} m)$ [Karger '96]

k-Lean Tree Decompositions and More

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  - Previously $k^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ [Cygan, Komosa, Lokshtanov, Pilipczuk, Pilipczuk, Saurabh, Wahlström '21]
  - and $k^{\mathcal{O}(k)} m^{1+o(1)}$ [Anand, Lee, Li, Long, Saranurak '24] (suboptimal unbreakability parameters)

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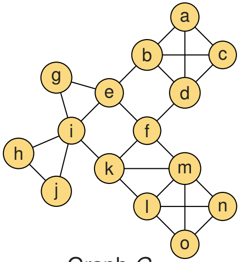
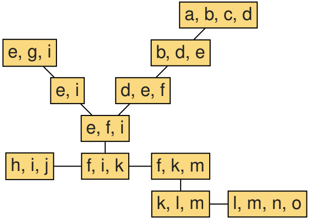


Graph G



A 3-lean tree decomposition of G

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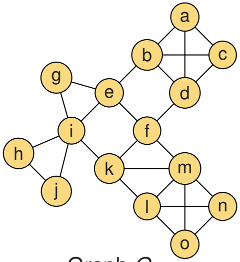

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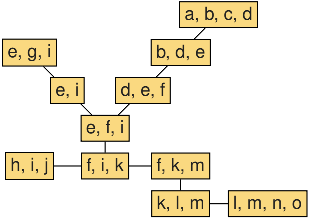

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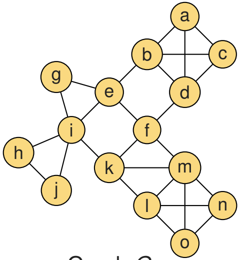

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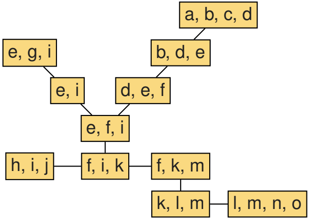

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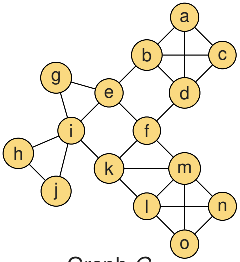

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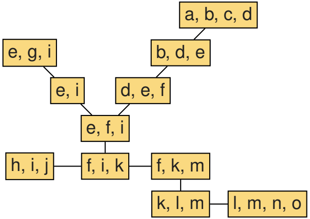

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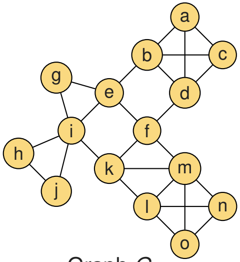

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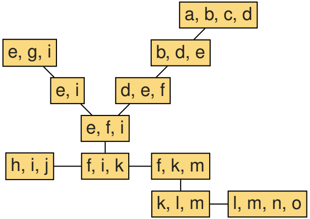

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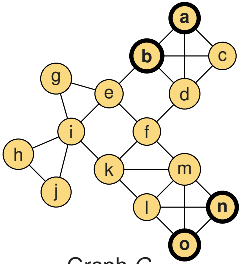

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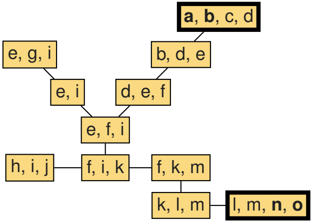

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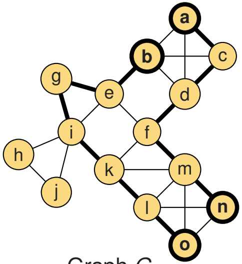

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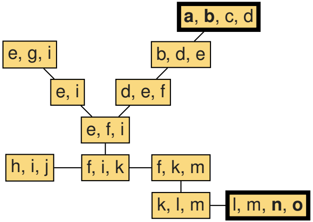

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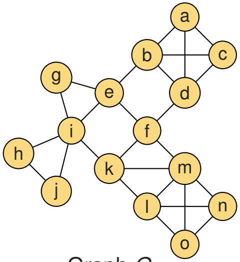

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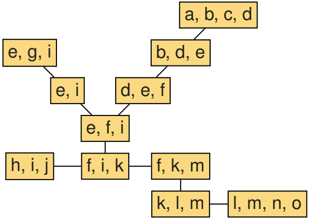

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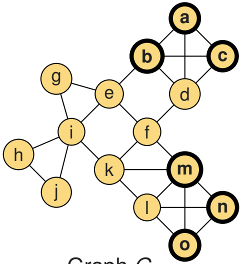

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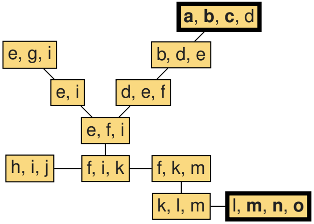

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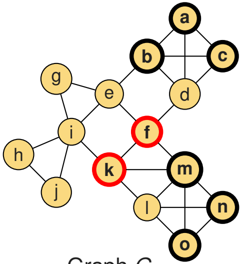

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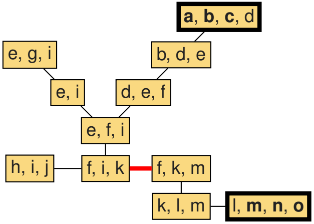

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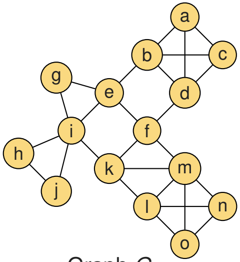

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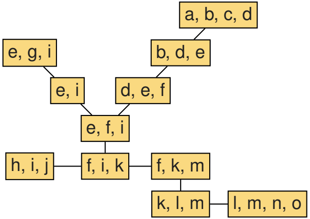
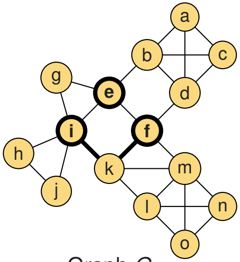


A 3-lean tree decomposition of G

- Tree decomposition:
  1. All vertices and edges are covered by bags
  2. For each vertex v, the bags containing v form a connected subtree
- k-lean:
  1. The adhesions (i.e. intersections of adjacent bags) have size $< k$
  2. For all pairs of bags B_1, B_2 and subsets $X_1 \subseteq B_1$, $X_2 \subseteq B_2$ with $|X_1| = |X_2| \leq k$, the sets X_1 and X_2 can be linked by vertex-disjoint paths if and only if there is no (B_1, B_2)-adhesion of size $< |X_1| = |X_2|$

k-Lean Tree Decompositions



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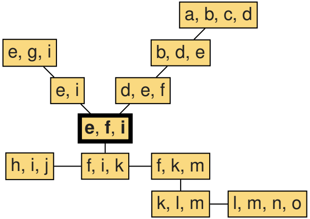

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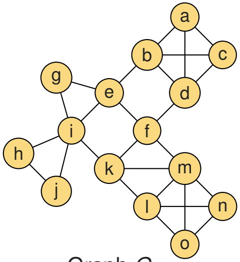

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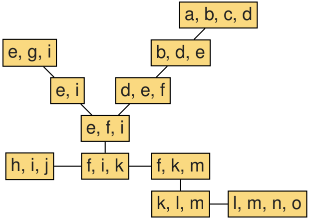

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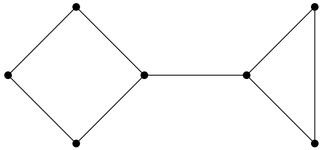
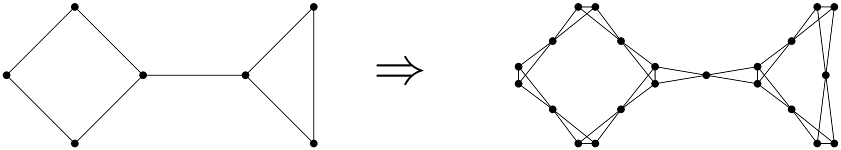
- Defined by [Thomas '90] (for $k = \infty$), and [Carmesin, Diestel, Hamann, and Hundertmark '14]

Reducing k-edge-connected components to k-lean tree decomposition



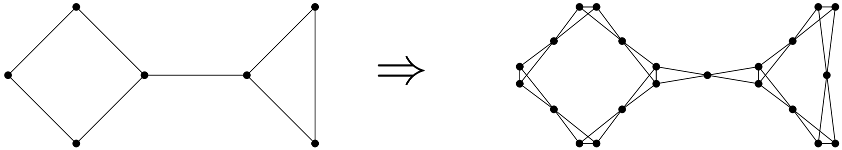
Reducing k-edge-connected components to k-lean tree decomposition

- Replace vertices by cliques of size k
- Create vertex for each edge and connect to the cliques corresponding to its endpoints



Reducing k-edge-connected components to k-lean tree decomposition

- Replace vertices by cliques of size k
- Create vertex for each edge and connect to the cliques corresponding to its endpoints
- Resulting k-lean tree decomposition gives a k-Gomory-Hu tree



Conclusion

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Main techniques:

- Recursive matching contraction compression (inspired by [Bodlaender'93])
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Thank you!

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Input: A "weakly-k-lean" tree decomposition:

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Lemma

If there is improver algorithm with running time $f(k) \cdot m$, then there is an algorithm that in time $k^{\mathcal{O}(1)} \cdot f(k) \cdot m$ computes a k-lean tree decomposition.

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Case 2: No matching of size $\Omega(n) \Rightarrow$ manage to recurse in some other way...

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