Fast FPT-Approximation of Branchwidth

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APGA 2022
May 3, 2022

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Improves approximation ratio from 3 to 2

Plan

1. Definitions
2. Overview of rankwidth algorithm
3. Combinatorial framework
4. Algorithmic framework

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Branchwidth

- Let V be a set and $f : 2^V \to \mathbb{Z}_{\geq 0}$ a symmetric set function.
  - Symmetric: For any $A \subseteq V$, it holds that $f(A) = f(\overline{A})$, where $\overline{A} = V \setminus A$

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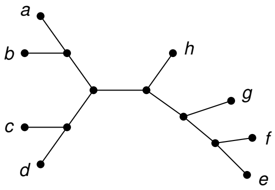
- Example with $V = \{a, b, c, d, e, f, g, h\}$:



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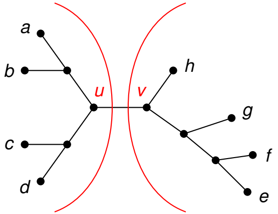

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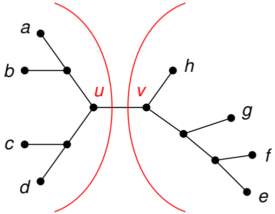

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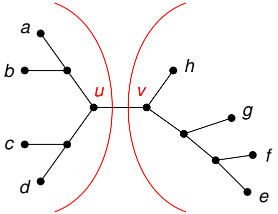

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- The width of the decomposition is $\max_{uv \in E(T)} f(uv)$
- The branchwidth of f is minimum width of a branch decomposition of f

Connectivity functions

- Function $f : 2^V \to \mathbb{Z}_{\geq 0}$ is a connectivity function if for any $A, B \subseteq V$:
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  - Rankwidth of G
- Also carving-width, matroid branchwidth, rankwidth in different fields...

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Input: Augmented rank decomposition of G of width k
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- Algorithmic framework:
  - Direct computation of refinements by dynamic programming $\rightarrow 2^{2^{\mathcal{O}(k)}} n^2$ time
  - Amortization techniques using combinatorial properties $\rightarrow 2^{2^{\mathcal{O}(k)}} n$ time

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General idea

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  - Let $f : 2^V \rightarrow \mathbb{Z}_{\geq 0}$ be a connectivity function
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- Strategy:
  - Let $h(T)$ be the number of edges of T of width k (heavy edges)
  - Either decrease $h(T)$ by using a **refinement operation**, or conclude that $k \leq 2\mathtt{bw}(f)$

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Specified by 4-tuple (r, C_1, C_2, C_3), where $r \in E(T)$ and (C_1, C_2, C_3) tripartition of V

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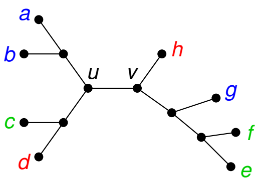

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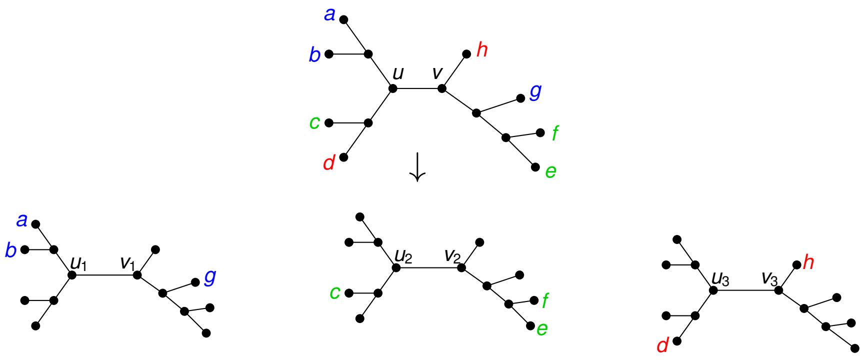

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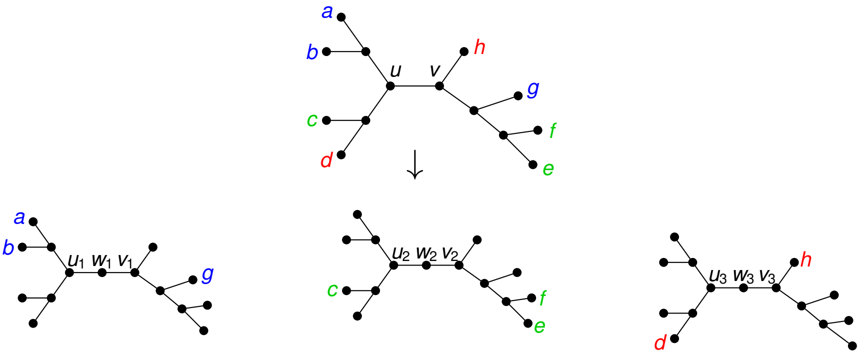

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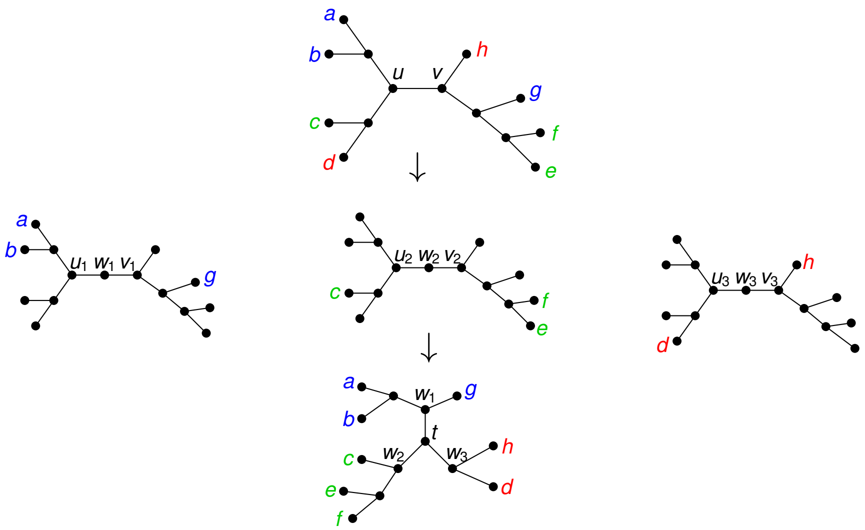

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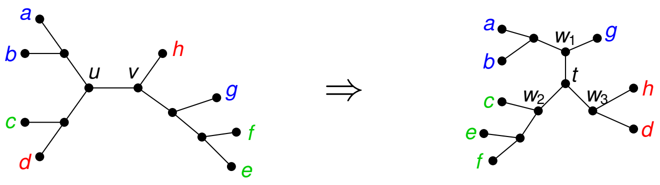

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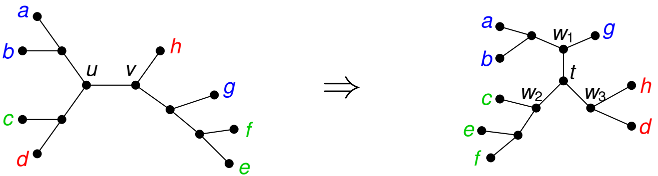

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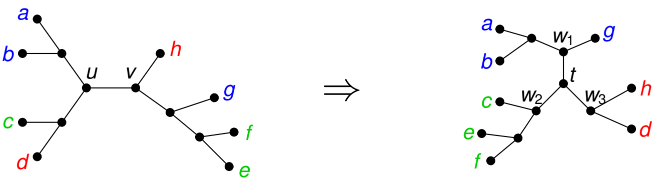

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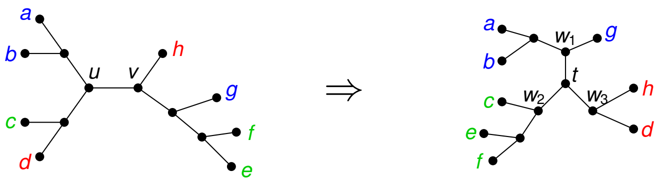

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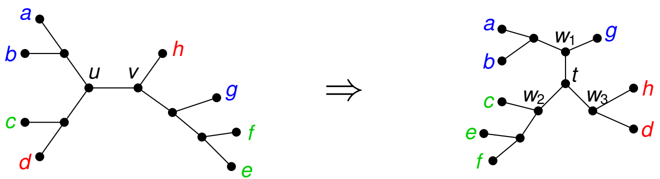

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- Observation 2: For each i, there will be edges corresponding to $(C_i \cap W, \overline{C_i \cap W})$ and $(C_i \cap \overline{W}, \overline{C_i \cap \overline{W}})$

Local Improvement

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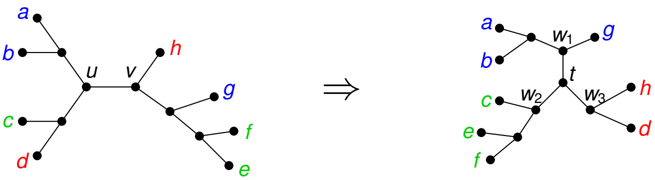

- Let $(W, \overline{W}) = (\{a, b, c, d\}, \{e, f, g, h\})$ be the cut of uv
- Combination of Observation 1 and 2:
  - The widths of edges "near the center" will be $f(C_i)$, $f(C_i \cap W)$, and $f(C_i \cap \overline{W})$ for each i

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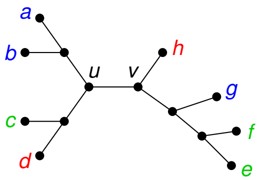
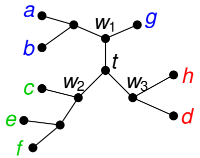


$\Rightarrow$



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For any set $W \subseteq V$ with $f(W) > 2\mathtt{bw}(f)$ there exists tripartition (C_1, C_2, C_3) of V so that for each i it holds that $f(C_i) < f(W)/2$, $f(C_i \cap W) < f(W)$, and $f(C_i \cap \overline{W}) < f(W)$.

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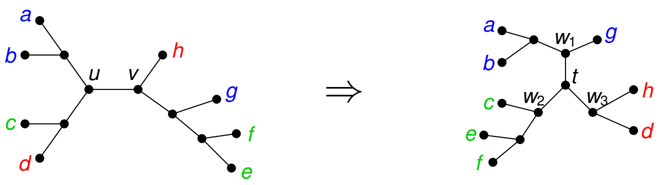

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$\Longrightarrow$ If $f(uv) > 2\mathrm{bw}(f)$, there exists refinement with uv that *"locally"* improves T

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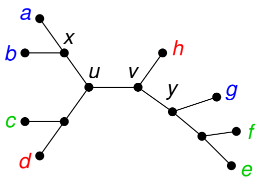
Theorem

If there exists a W-improvement, then there exists a W-improvement (C_1, C_2, C_3) so that refinement with (uv, C_1, C_2, C_3) does not increase width and decreases the number of heavy edges.

Global Improvement: Observation



- Consider T rooted at $r = uv$
- For a node $x \in V(T)$, denote by $T_r[x] \subseteq V$ the leaves in the subtree below x
  - Example: $T_r[x] = \{a, b\}$ and $T_r[y] = \{e, f, g\}$

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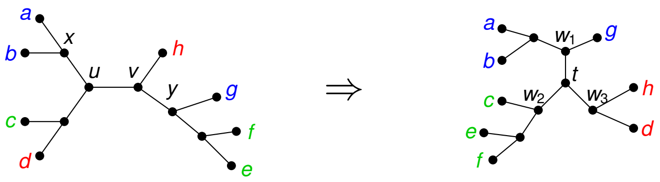

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- For a node $x \in V(T)$, denote by $T_r[x] \subseteq V$ the leaves in the subtree below x
  - Example: $T_r[x] = \{a, b\}$ and $T_r[y] = \{e, f, g\}$
- Let T' be refinement of T with (r, C_1, C_2, C_3)
- Observation: Each edge of T' corresponds either to $(C_i, \overline{C_i})$ or to $(T_r[x] \cap C_i, \overline{T_r[x] \cap C_i})$ for some $x \in V(T)$

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  $\Longrightarrow$ Strict improvement

Plan

1. Definitions
2. Overview of rankwidth algorithm
3. Combinatorial framework
4. **Algorithmic framework**

First Algorithm

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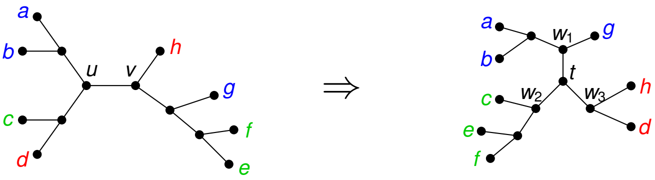

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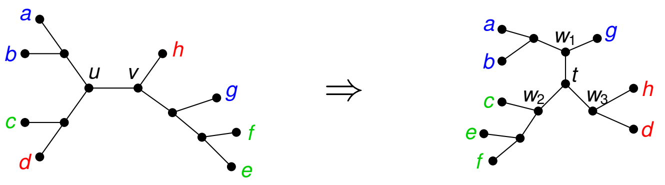

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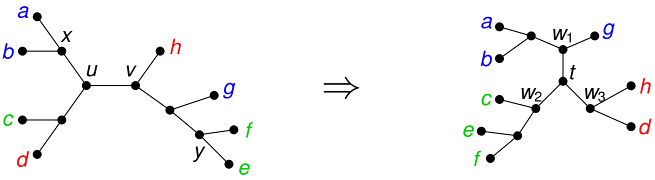

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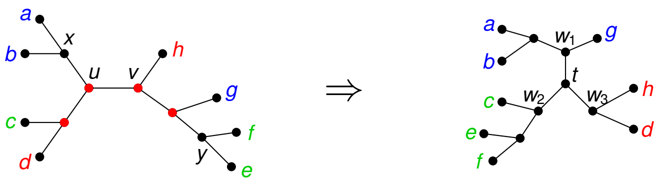

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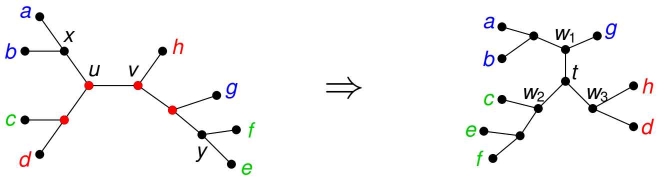

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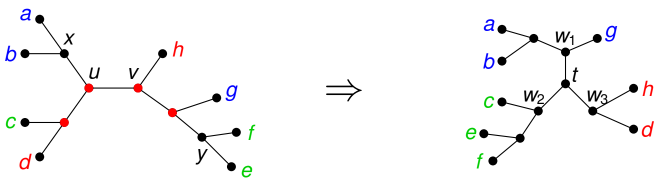

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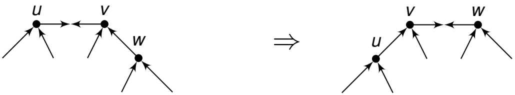

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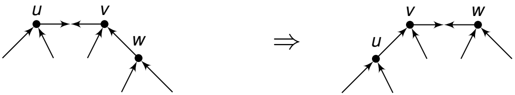

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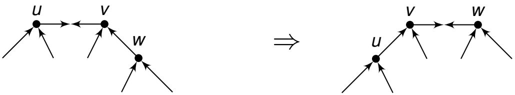

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Bibliography