Minor Containment and Disjoint Paths in almost-linear time

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based on joint work with Michał Pilipczuk and Giannos Stamoulis
from the University of Warsaw (accepted to FOCS 2024)

19 September 2024

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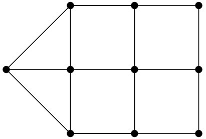

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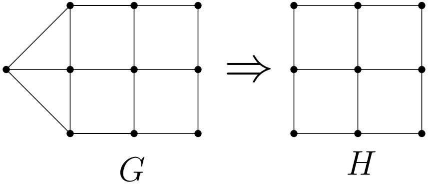

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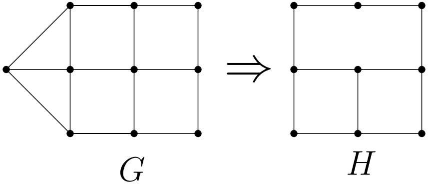

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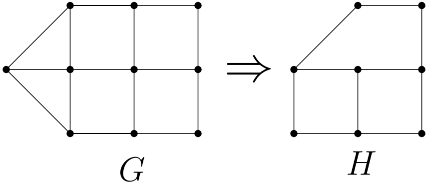

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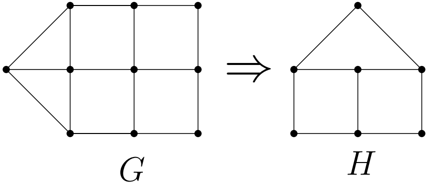

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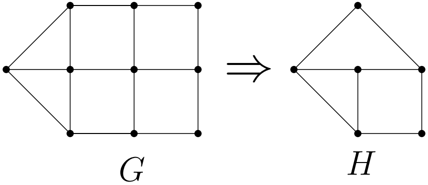

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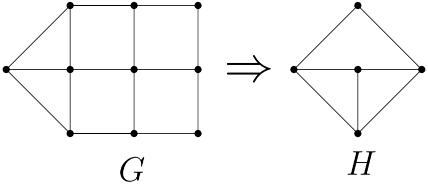

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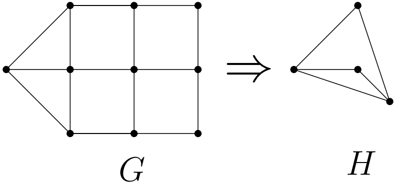

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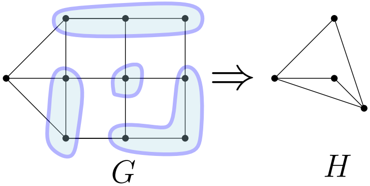

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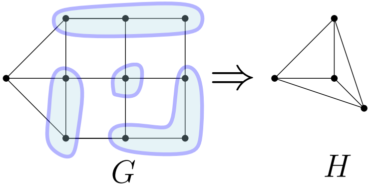

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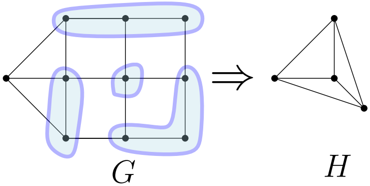

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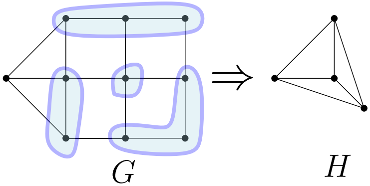

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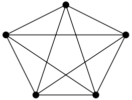
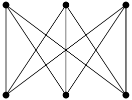
Planar $\Leftrightarrow$ no  or 

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The algorithm

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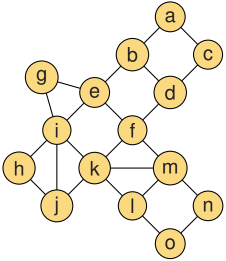
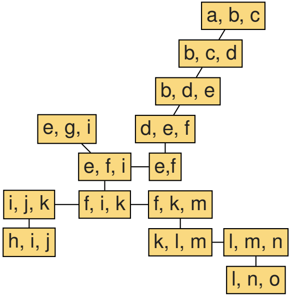


Graph G
Treewidth 2



A tree decomposition of G
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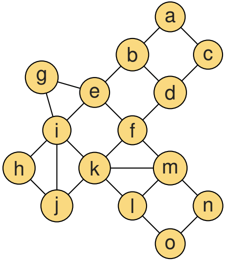

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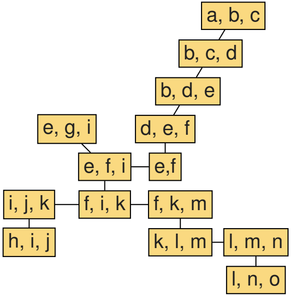

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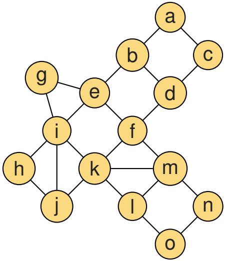

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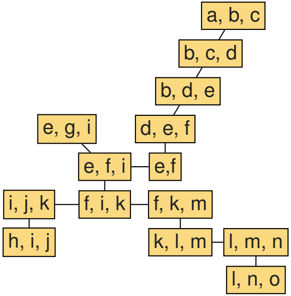

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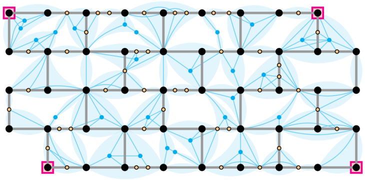

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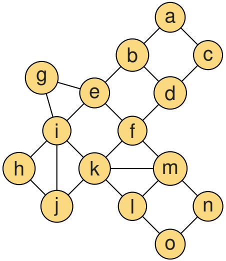

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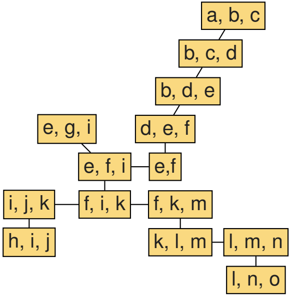

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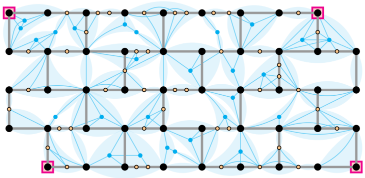

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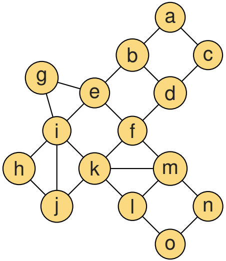

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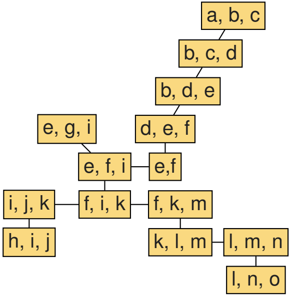

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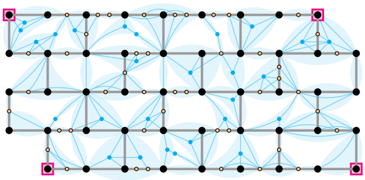

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1. Fast implementation of the irrelevant vertex technique on apex-minor-free graphs

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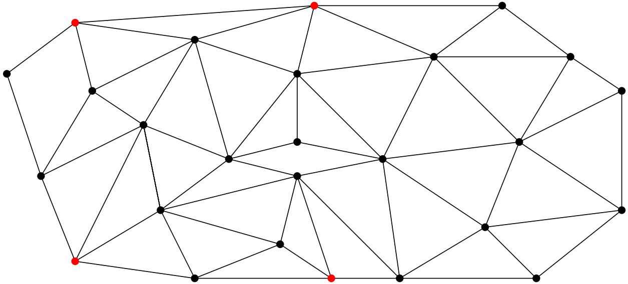
4. Reducing general graphs to clique-minor-free graphs

Algorithm for apex-minor-free graphs



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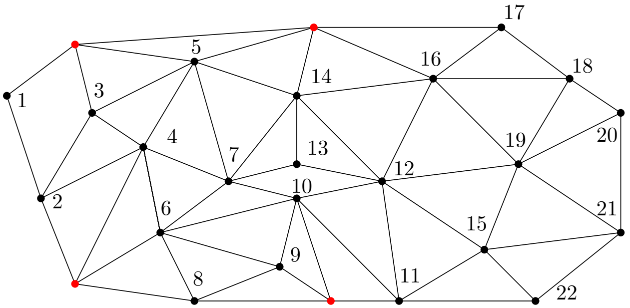

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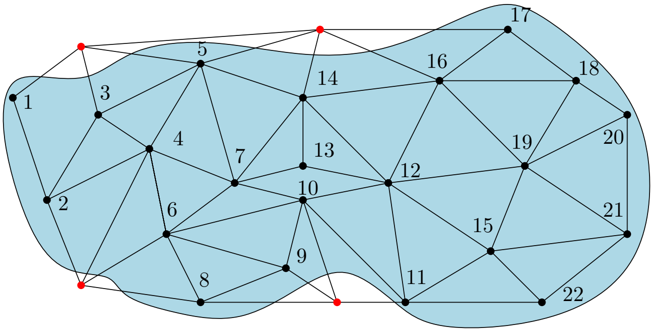

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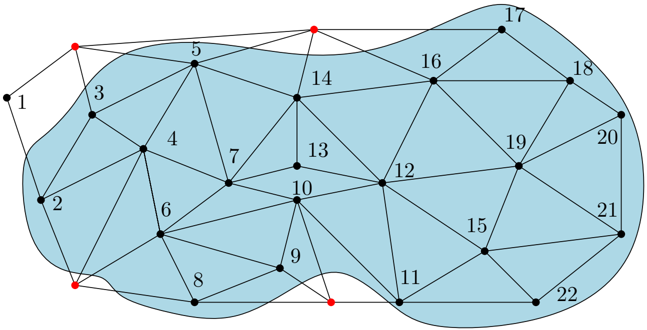

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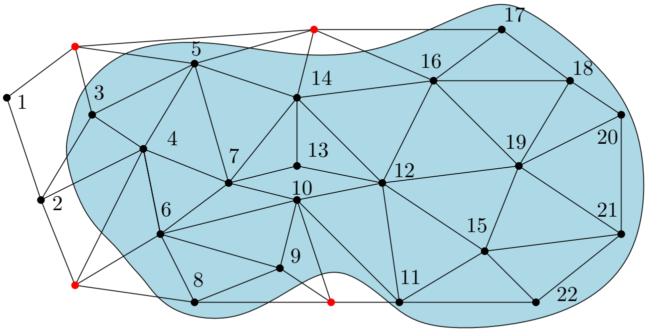

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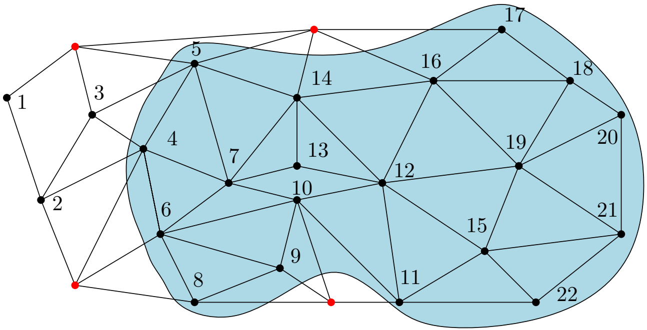

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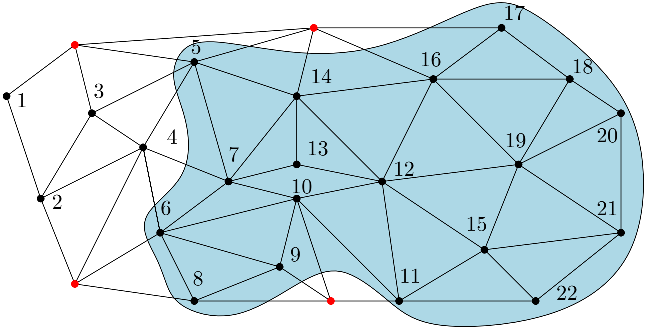

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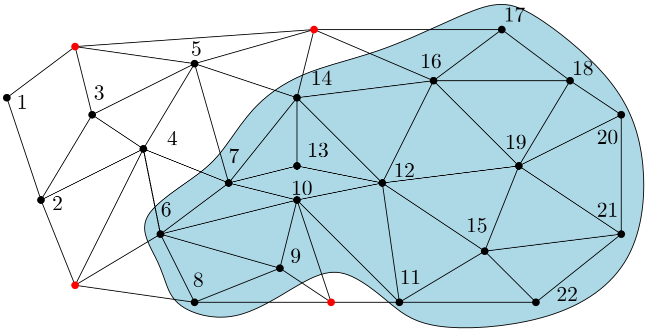

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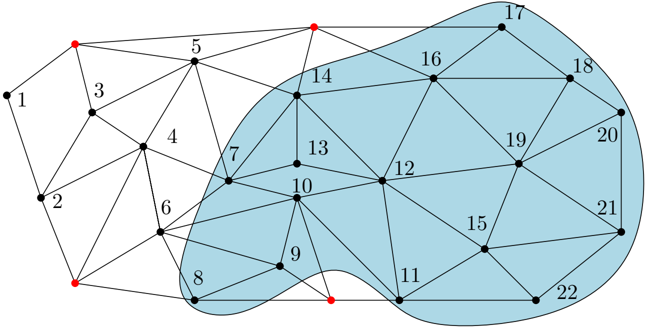

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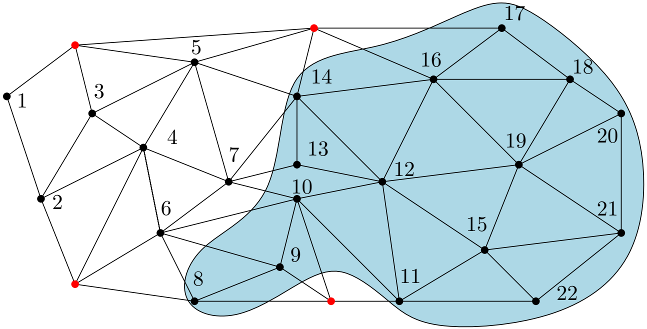

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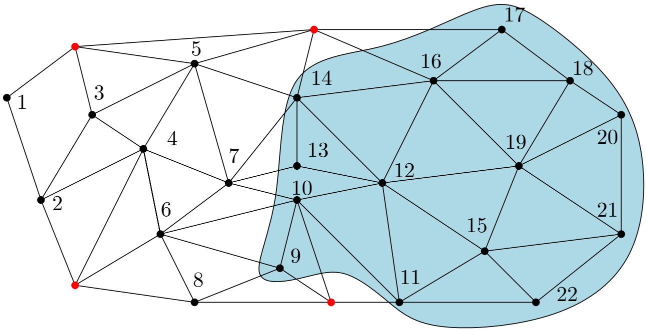

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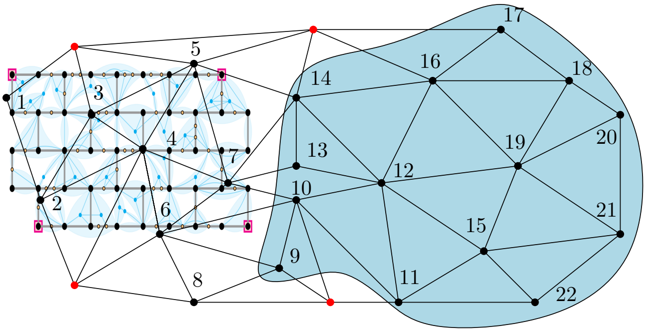

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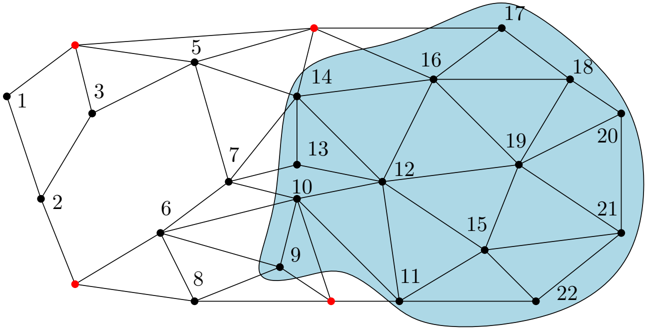

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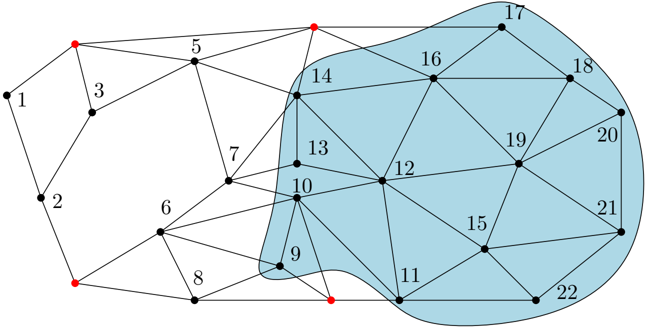

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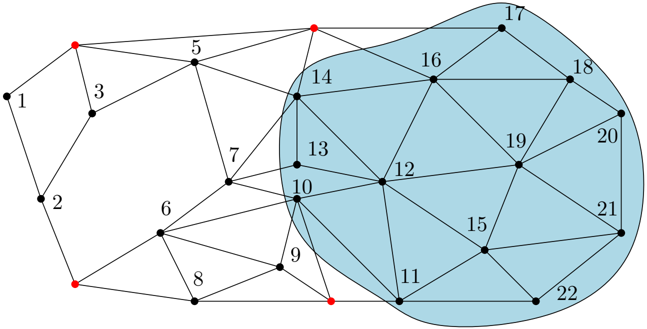

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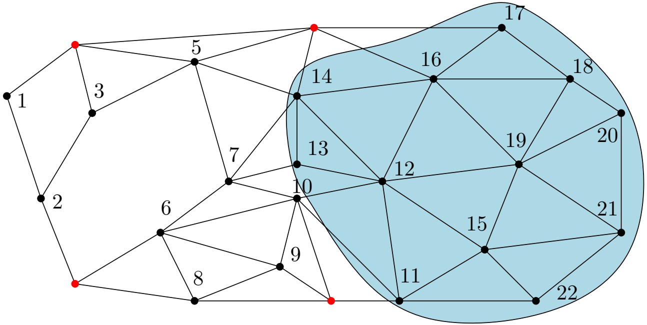

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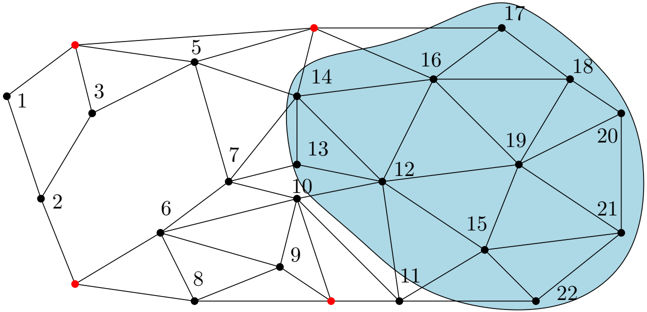

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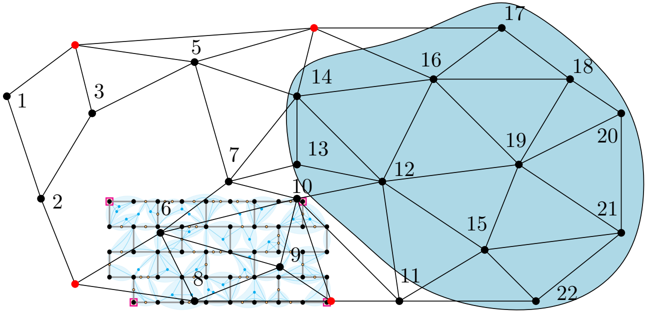

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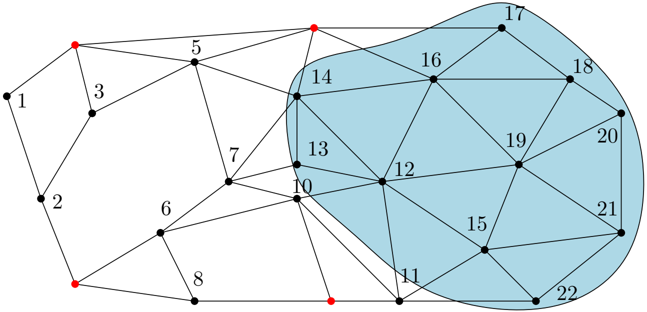

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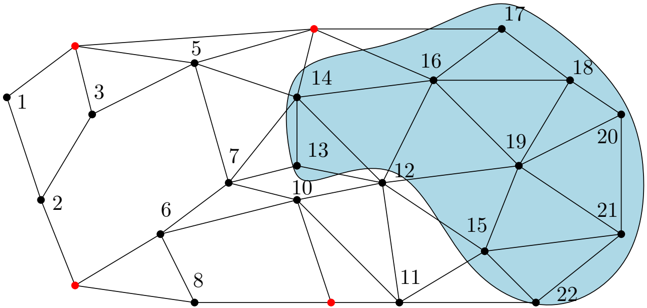

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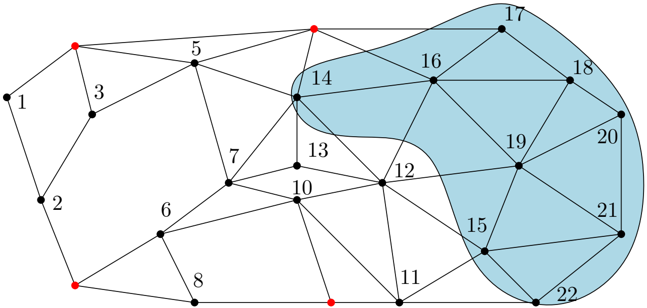

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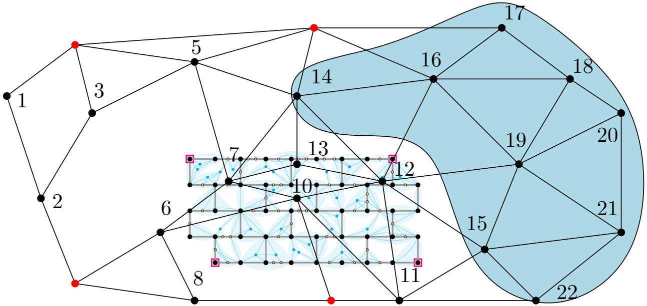

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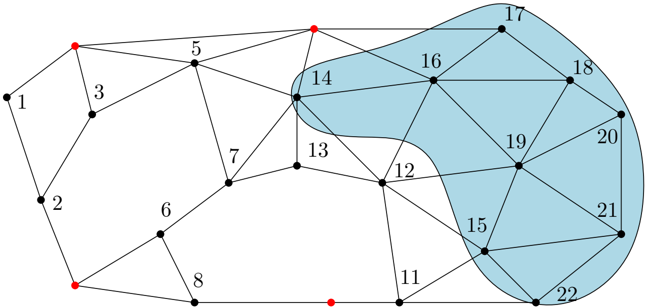

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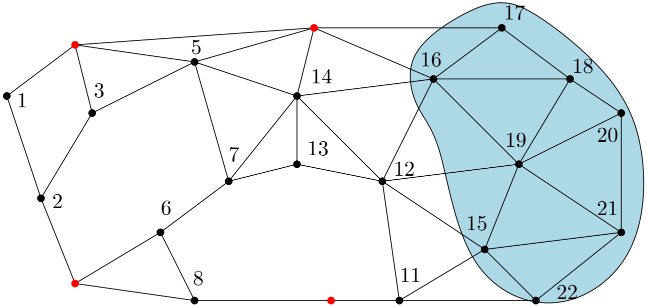

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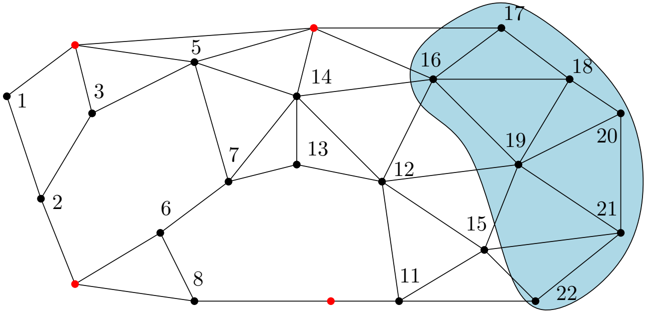

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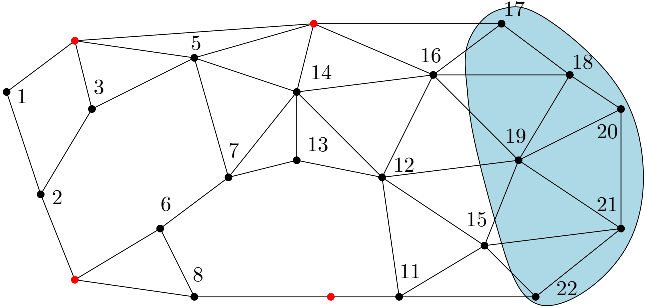

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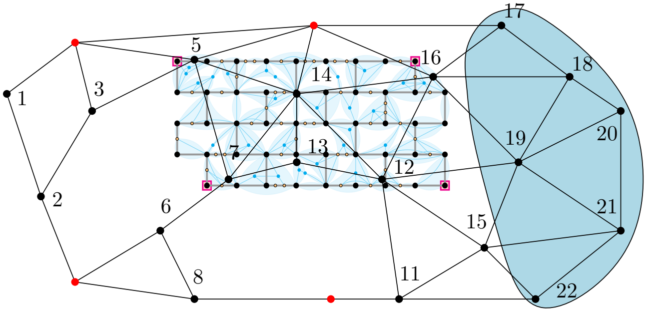

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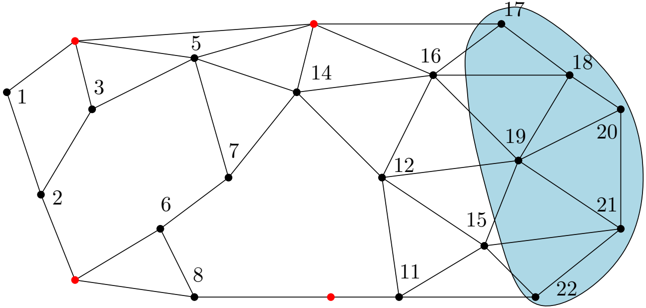

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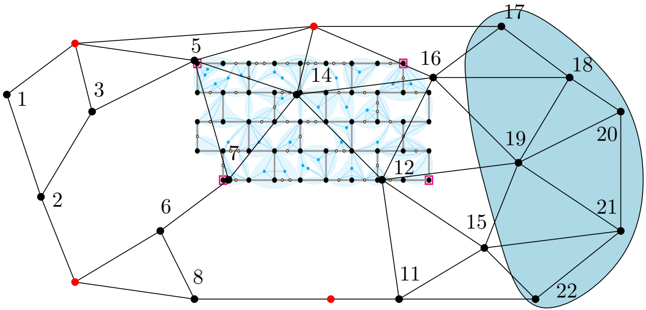

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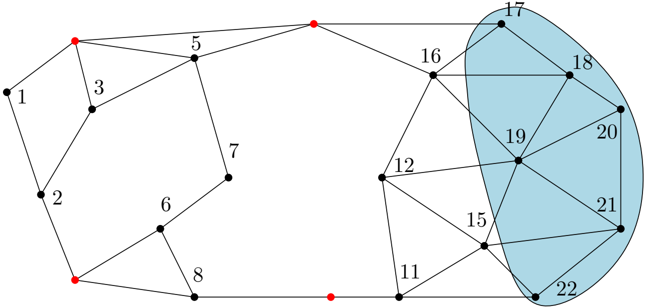

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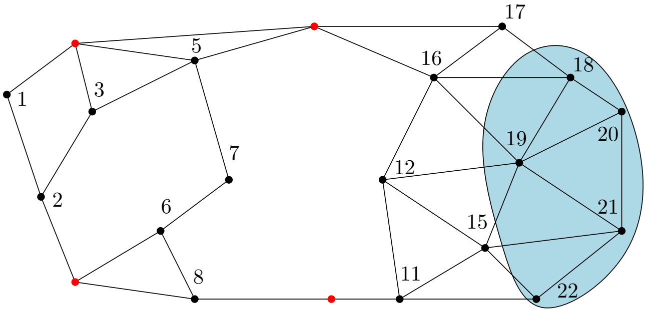

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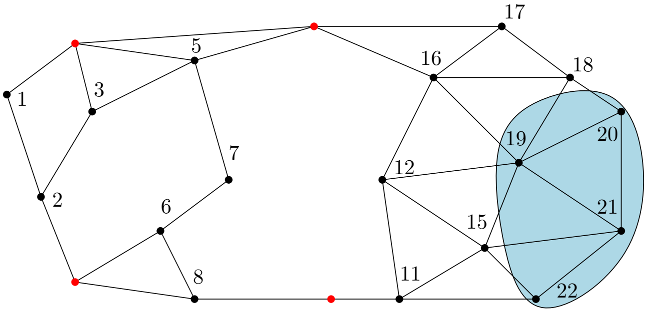

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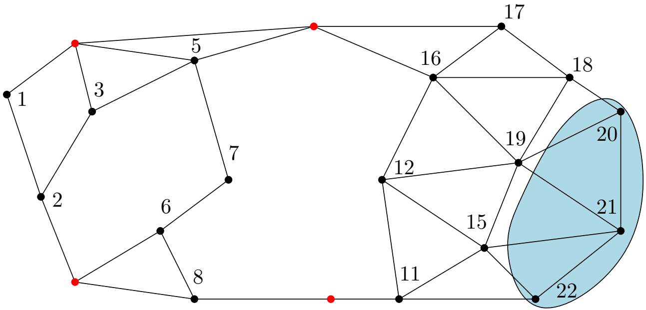

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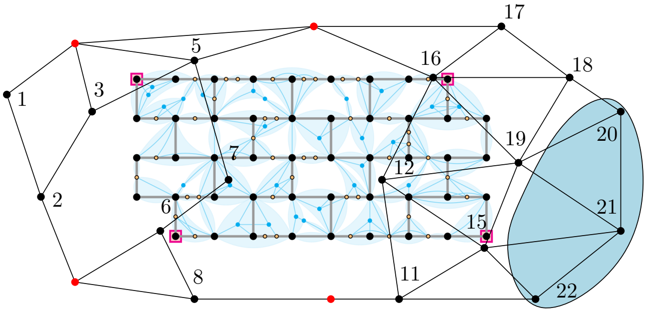

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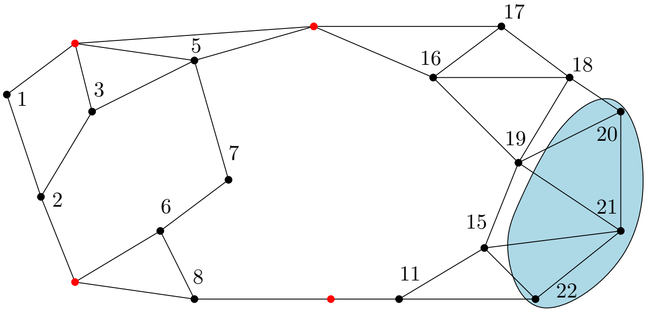

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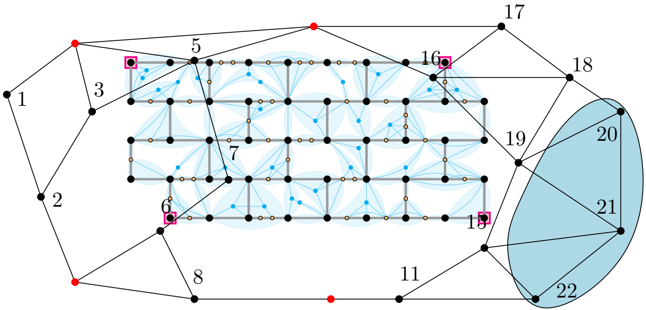

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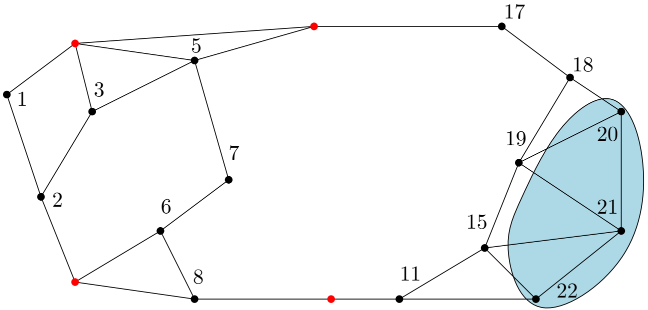

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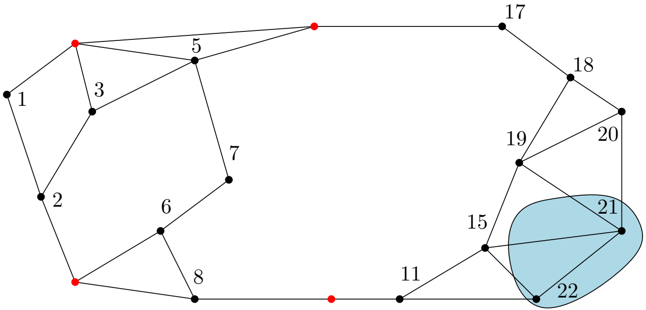

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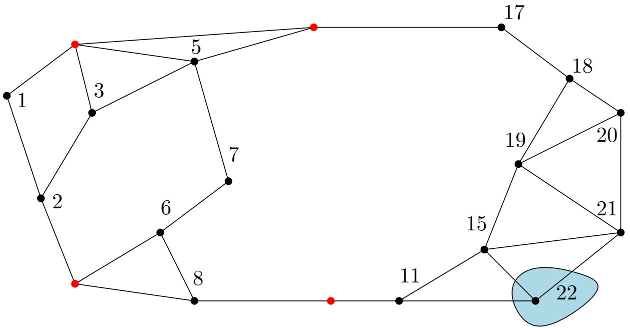

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Thank you!