How Graph Structure Makes Hard Problems Tractable

Tuukka Korhonen

1 April 2026

CS Special Seminar

Role of Structure in Algorithms

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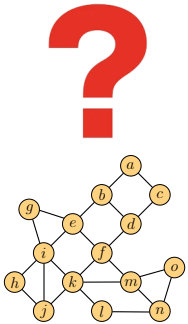

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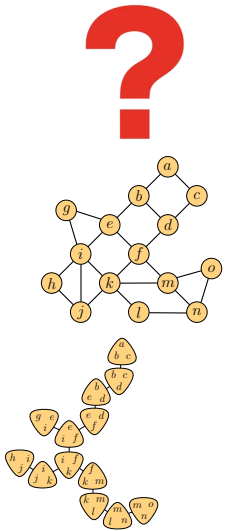

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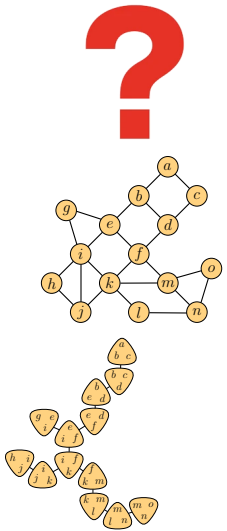
1. Drawing the line between easy and hard instances using graph-theoretic methods
2. Designing algorithms to discover the graph-theoretic structure that can be exploited



Outline of the Talk

⇒ Background and motivation

– My contributions

– Future research

Example: Constraint Satisfaction Problems (CSPs)

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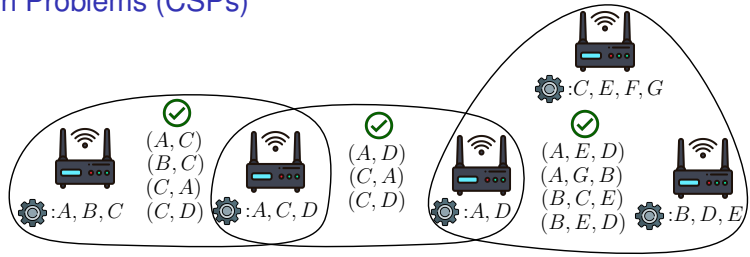
Wifi setting assignment:

Variables: Transmitters

Domains: Settings of transmitters

Constraints: Compatible settings



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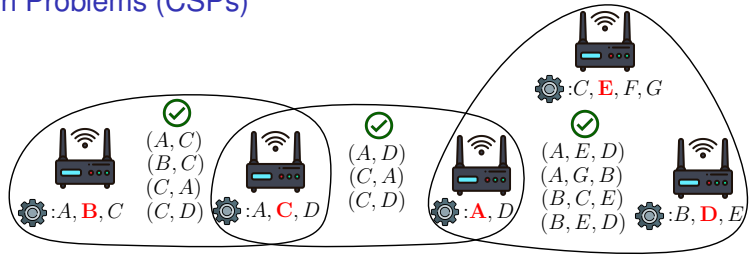

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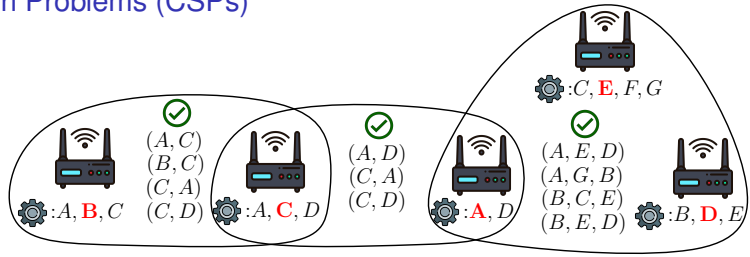
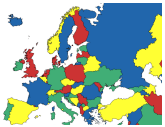


Graph coloring:

Variables: Vertices

Domains: Possible colors

Constraints: Endpoints of each edge get different colors



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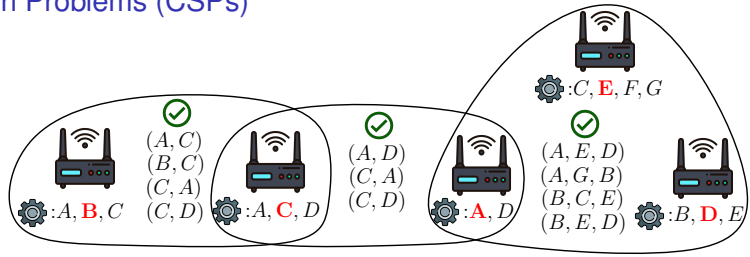

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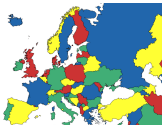

Crosswords:

Variables: Squares

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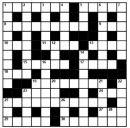
Constraints: Letters must form words in the dictionary



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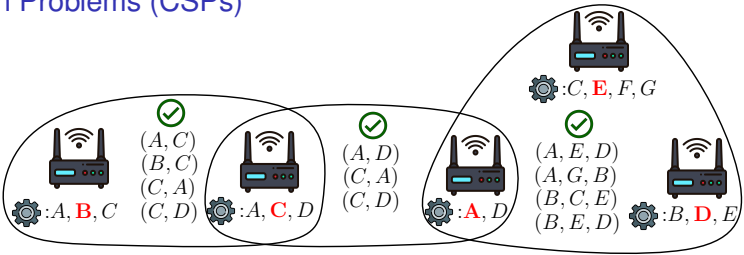

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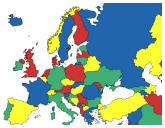

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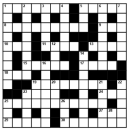

Join evaluation in databases:

Variables: Attributes (columns)

Domains: Values of attributes

Constraints: Tables of the database

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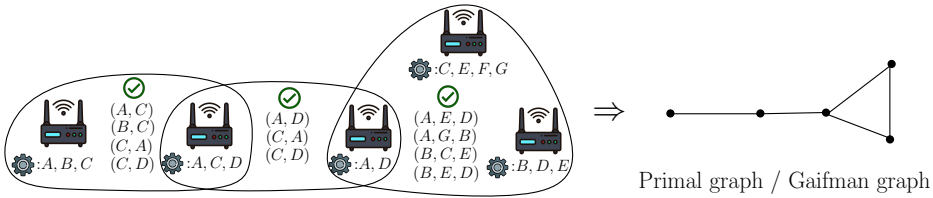

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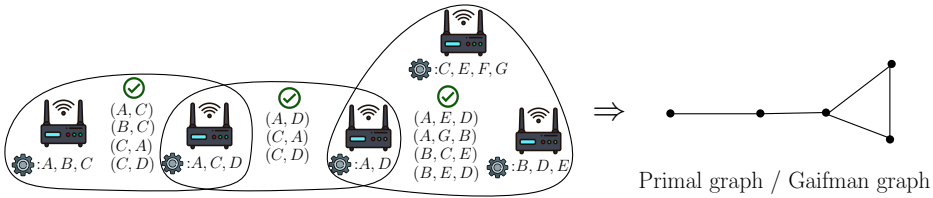
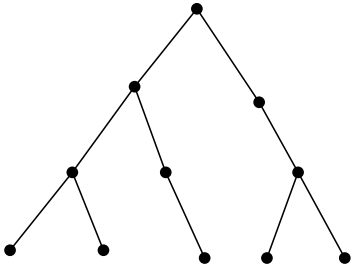


Warmup: What if the primal graph is a tree?



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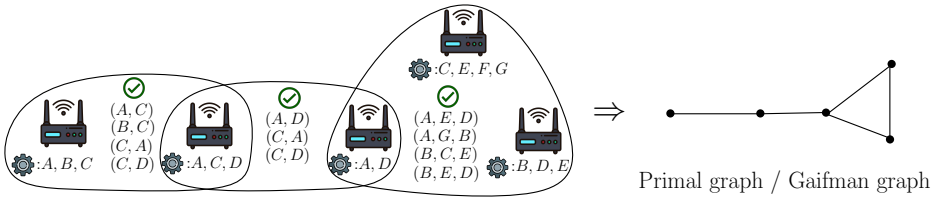

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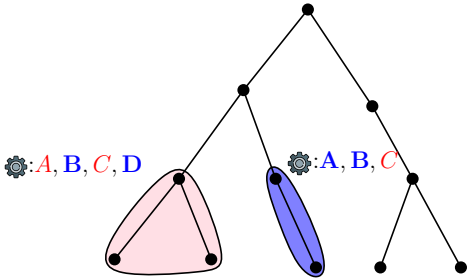

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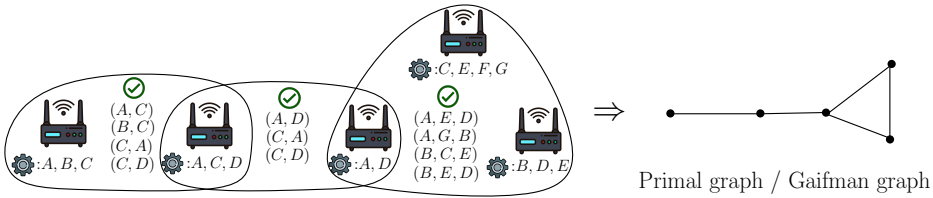

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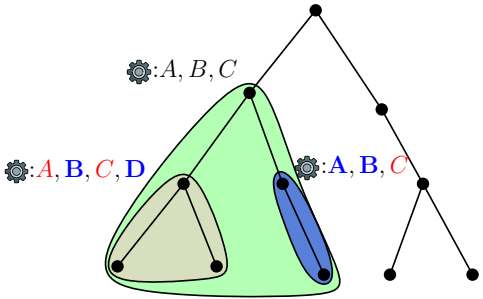

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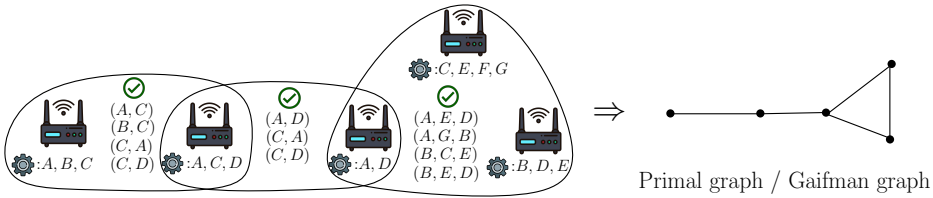

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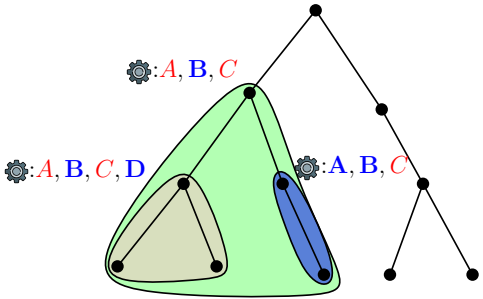
⇒ Linear-time algorithm



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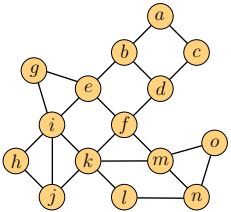

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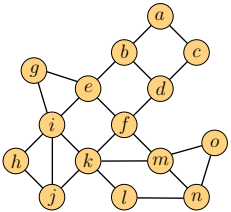

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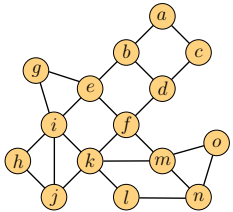
- The example graphs have treewidth 2
- Cliques have maximum treewidth, $n - 1$



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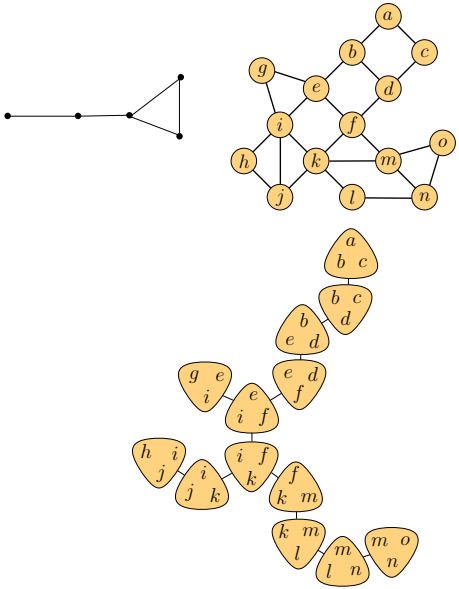
Width of a tree decomposition: Maximum bag size -1



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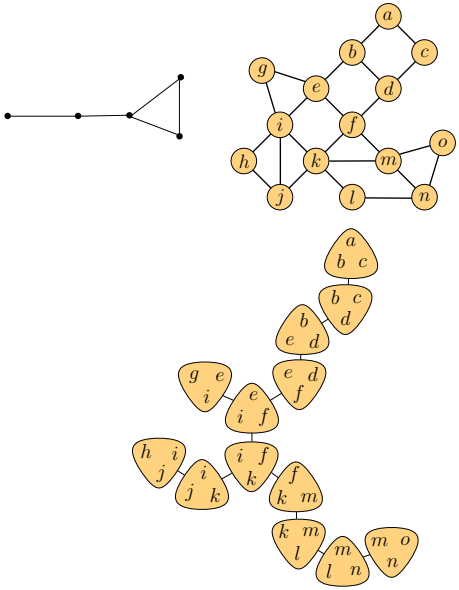

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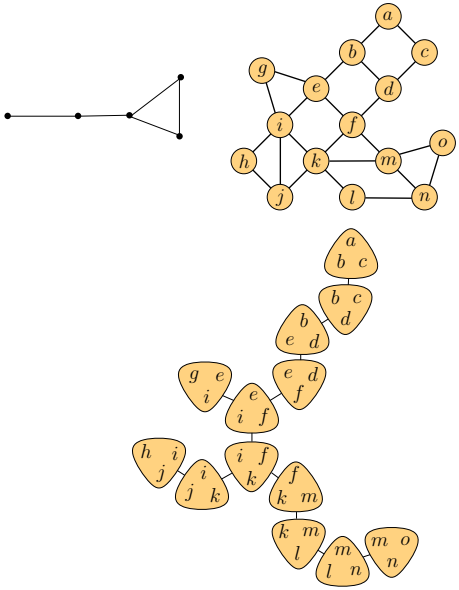

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Instance-optimal over the primal graphs

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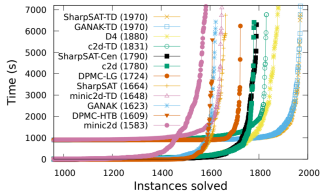

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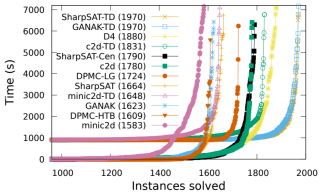

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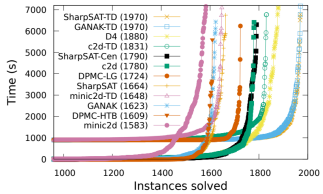

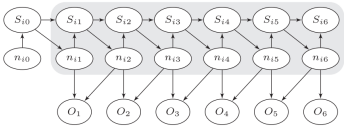

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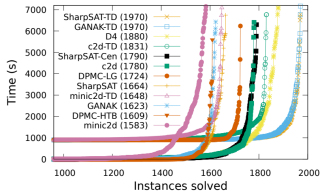
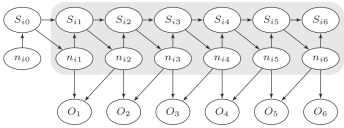
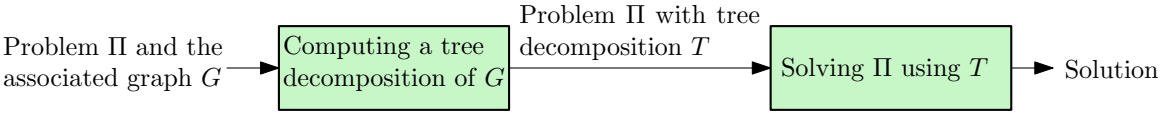
- Join evaluation in databases (fractional hypertreewidth, submodular width)





Computing Treewidth



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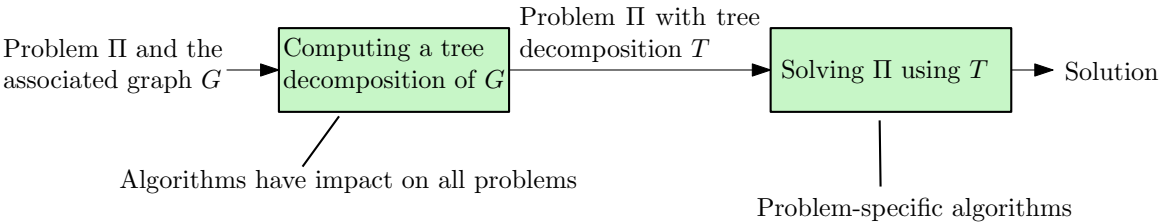

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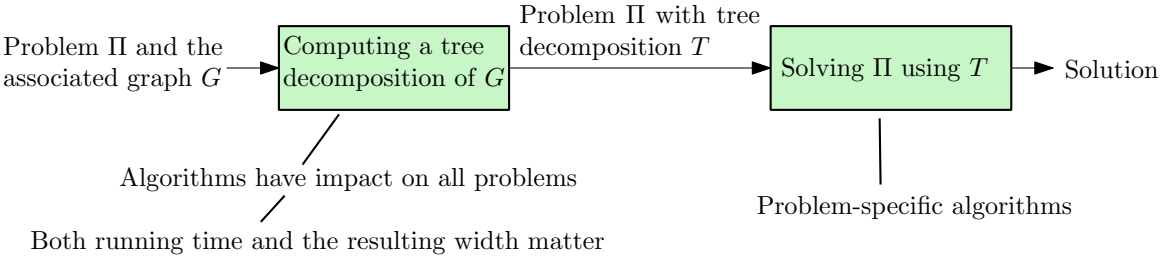

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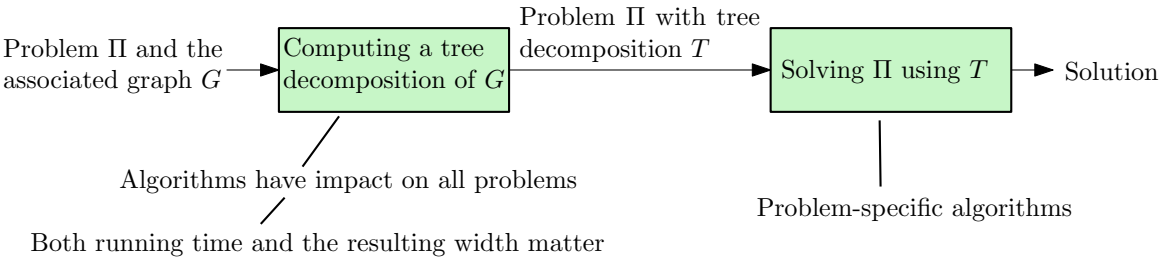

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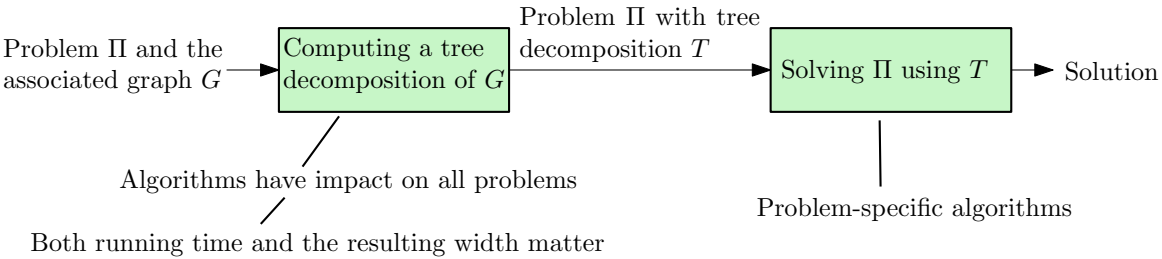

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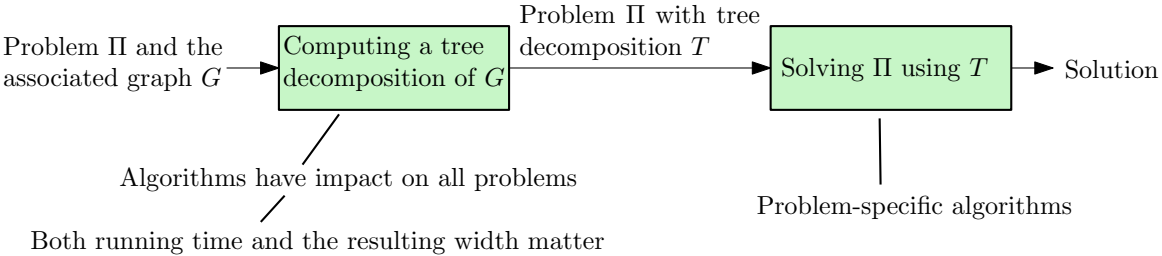

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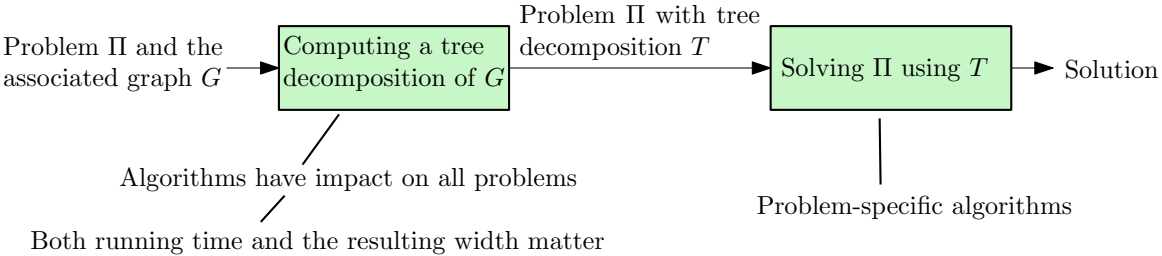

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- [Robertson & Seymour, 1987]: Treewidth can be 4-approximated in time $\mathcal{O}(3^{3k}n^2)$

Outline of the Talk

– Background and motivation

⇒ My contributions

– Future research

Algorithms for Computing Treewidth

Ref.	Appx. ratio	Run. time
[RS'87]	4	$3^{3k} n^2$
[ACP'87]	exact	n^{k+1}
[RS'87]	exact	$f(k) \cdot n^2$
[Lag'90]	8	$k^{\mathcal{O}(k)} n \log^2 n$
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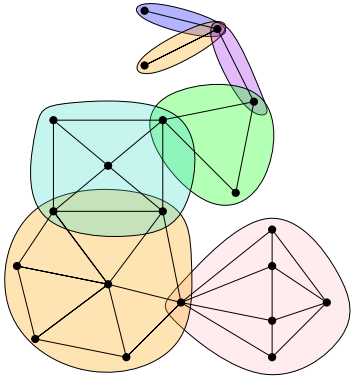
  - Partially solves problem of [Bodlaender'93] about dynamic treewidth

Beyond Treewidth

- In many settings, we wish to decompose a graph until remaining parts are “well-connected”



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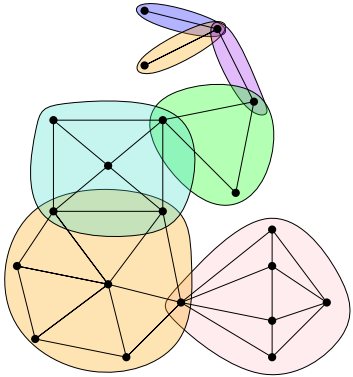

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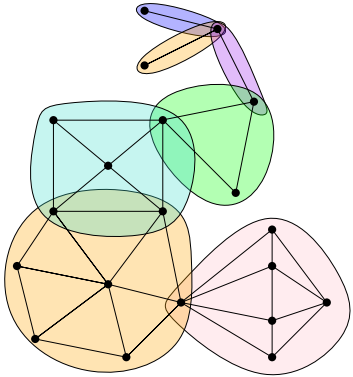

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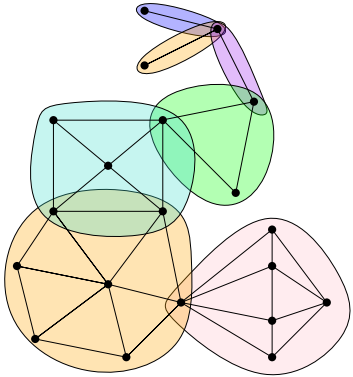

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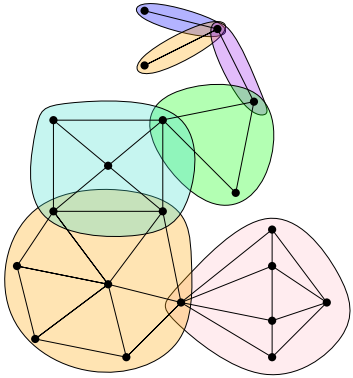


- [Korhonen, FOCS’25]: Adapted the techniques to give an optimal running time for dynamic treewidth

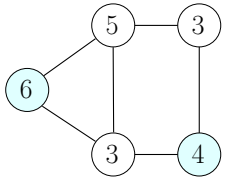
Outline of the Talk

– Background and motivation

– My contributions

⇒ Future research

Understanding Tractability of Maximum Weight Independent Set Problem (MWIS)



$\Rightarrow \quad 10$

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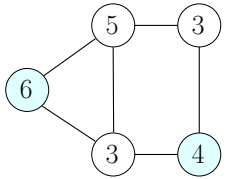

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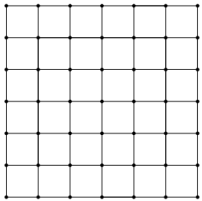

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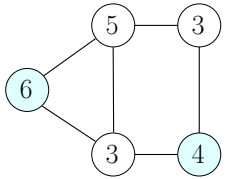

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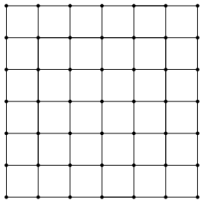

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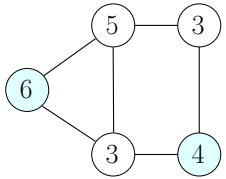

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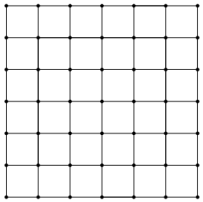

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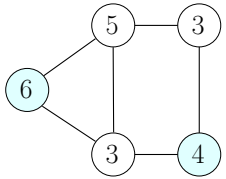

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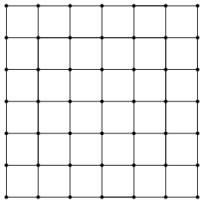

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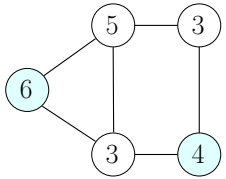

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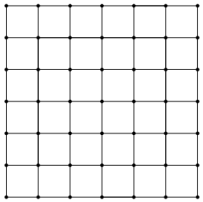

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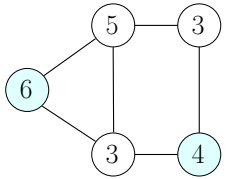

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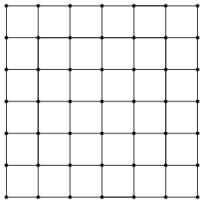

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- Conjecture wide open, but progress on special cases, e.g. [Bonamy, Bonnet, Déprés, Esperet, Geniet, Hilaire, Thomassé, Wesolek, SODA'23], [Korhonen & Lokshtanov, SODA'24], [Chudnovsky, Gartland, Hajebi, Lokshtanov, Spirkl, SODA'25]

Fully Polynomial-Time Parameterized Graph Decomposition

Constant-factor approximation

$2^{\mathcal{O}(k)}n$ running time

Treewidth, k-unbreakable tree decomposition...

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Potential techniques:

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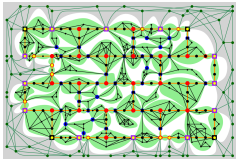
Understanding High Treewidth via Constant-Congestion Minors



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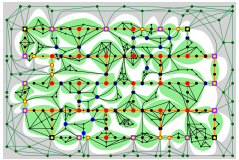

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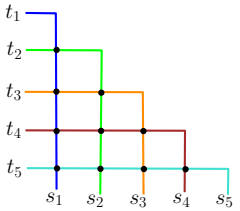

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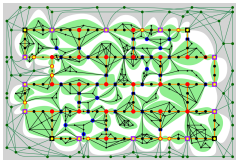

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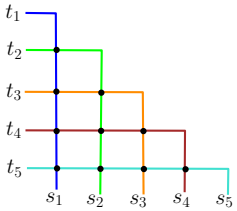

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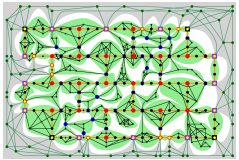

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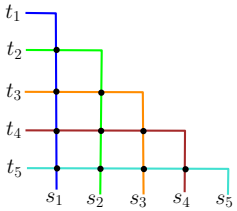

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Concrete settings:

- Grohe-Schwentick-Segoufin-Marx CSP result for bounded-domain CSPs?
- SAT?
- Monadic Second-Order logic?

Conclusion

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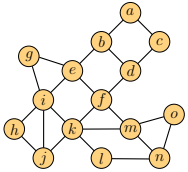

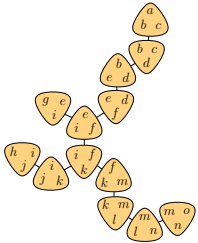

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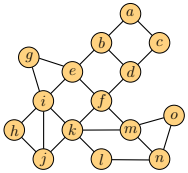

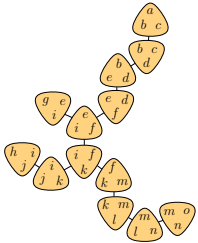

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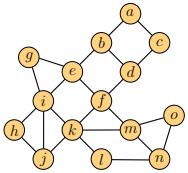

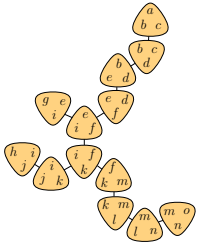

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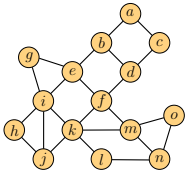

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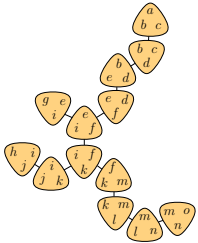

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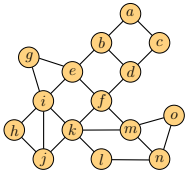

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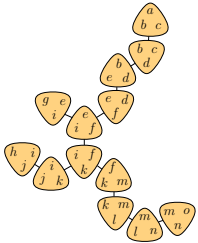

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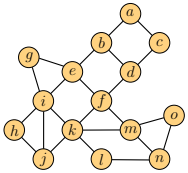

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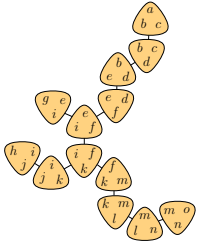

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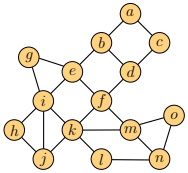

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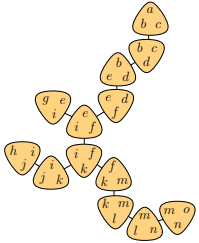

Thank you!