Stability in Graphs with Matroid Constraints

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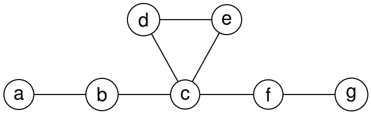
12 June 2024

Frameworks

- Framework is a pair (G, M), where
  - $G = (V, E)$ is an undirected graph



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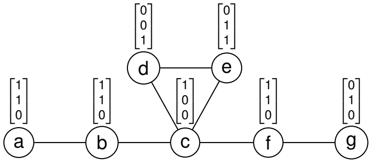

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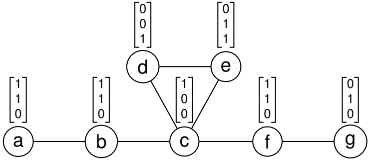

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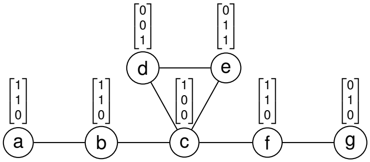

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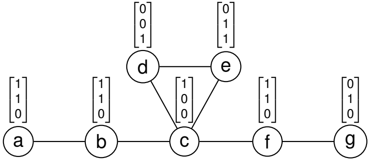

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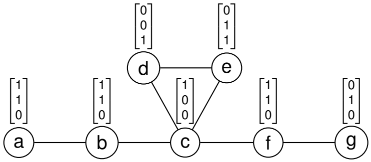

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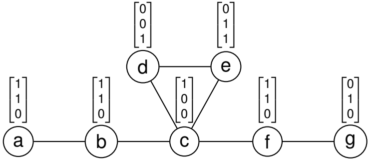

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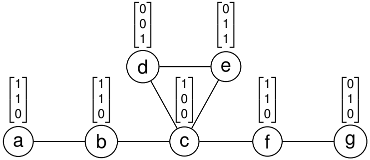

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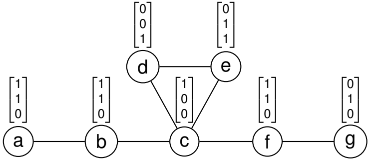

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- This work: M is either given by an independence oracle or represented by a matrix

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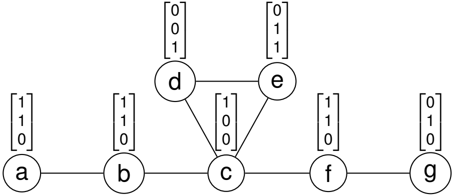

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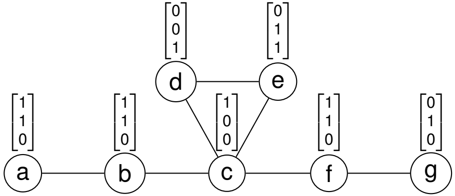

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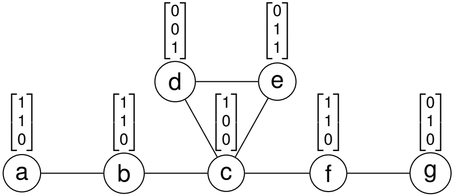

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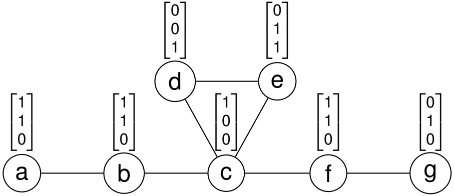

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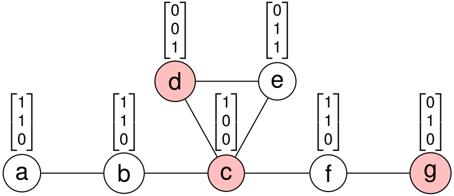

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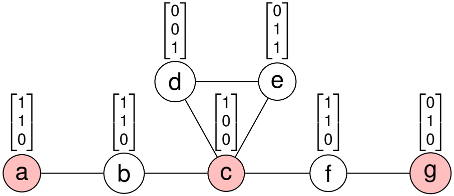

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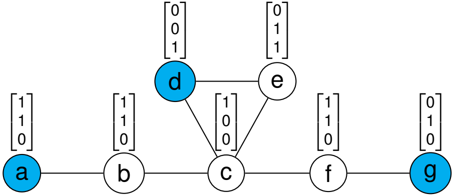

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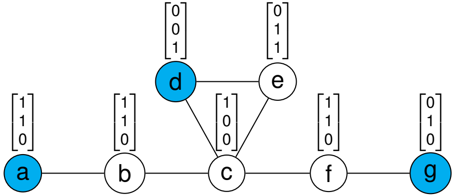

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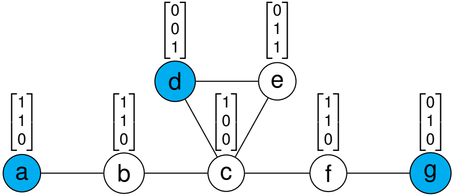

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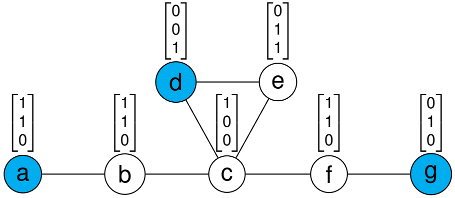

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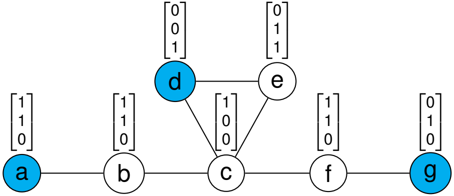

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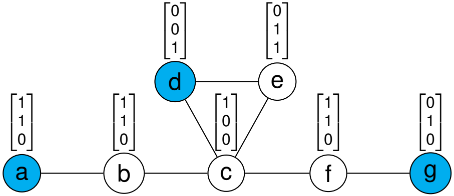
- Transversal matroid
  - Bipartite matching with separation [Manurangsi, Segal-Halevi, Suksompong '23]



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When M is given by independence oracle, no $f(k) \cdot n^{o(k)}$ time algorithm (unconditionally), even on

- bipartite graphs
- chordal graphs
- claw-free graphs
- AT-free graphs

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No polynomial kernel parameterized by $k + d$, unless NP $\subseteq$ coNP$/$poly

FPT algorithm parameterized by k, d

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For every vertex v, exists an optimal solution that intersects $N[v]$

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Lemma

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Step 2: Pick a vertex v with degree $\leq d$ and branch

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Assume M truncated to rank k

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$\Rightarrow k^4$ size kernel

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- This work:
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Thank you!

Linear matroids

Recall: No $f(k) \cdot n^{o(k)}$ algorithm on chordal graphs when M given by independence oracle

Theorem (This work)

There is a $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ time algorithm for Independent Stable Set when G is chordal and M is a linear matroid given by its representation.

(but no polynomial kernel, unless NP $\subseteq$ coNP/poly)

Idea:

- Dynamic programming over tree decomposition with representative sets

In contrast, Independent Stable Set is W[1]-hard when G is bipartite and M is linear