Induced-Minor-Free Graphs: Separator Theorem, Subexponential Algorithms, and Improved Hardness of Recognition

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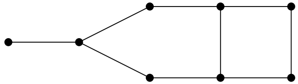
Graph containment



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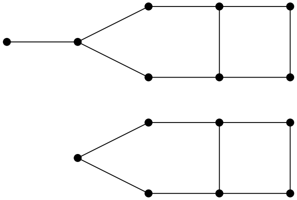

1. Induced subgraph
   - vertex deletions

Graph containment

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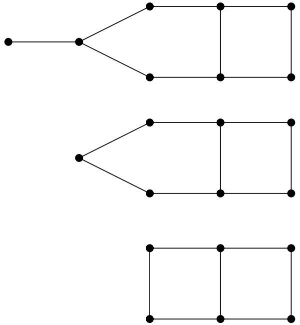
2. Induced minor
   - vertex deletions
   - edge contractions



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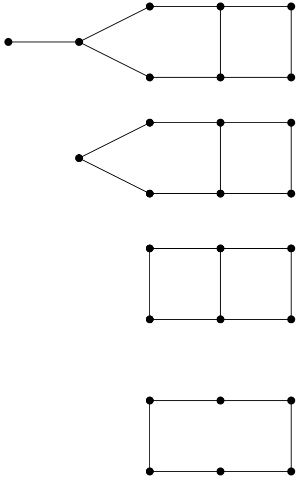

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3. Minor
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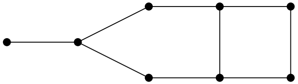

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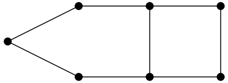

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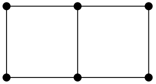

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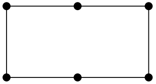

In this talk, all graphs are simple! (no self loops or parallel edges)

Graph classes defined by containment

- For a graph H, we can define graph classes by excluding H
  - H-minor-free graphs
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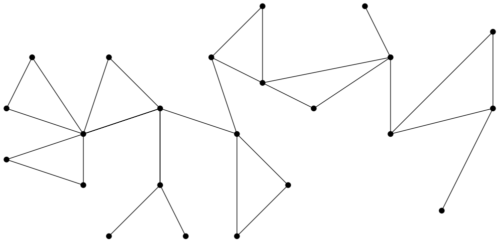
- Example: C_4-minor-free graphs



- Every biconnected component is a triangle
- Chordal and treewidth ≤ 2

Induced-minor-free graphs

How about C_4-induced-minor-free graphs?

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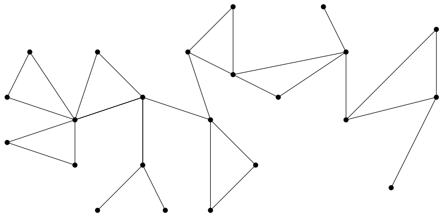
- Contain all C_4-minor-free graphs



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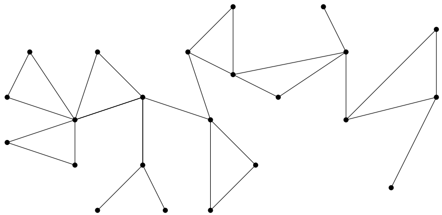
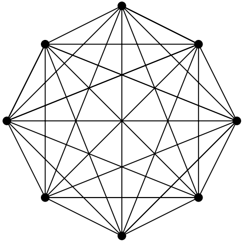
How about C_4-induced-minor-free graphs?

- Contain all C_4-minor-free graphs
- Contain all cliques





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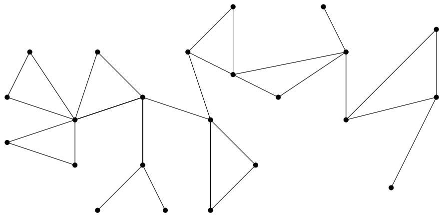
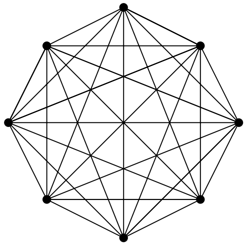
- Contain all C_4-minor-free graphs
- Contain all cliques ⇒ unbounded treewidth





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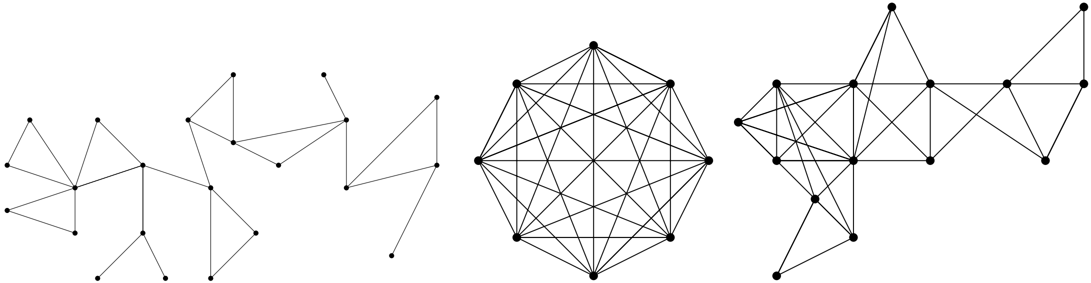
- Contain all C_4-minor-free graphs
- Contain all cliques $\Rightarrow$ unbounded treewidth
- C_4-induced-minor-free graphs $=$ chordal graphs



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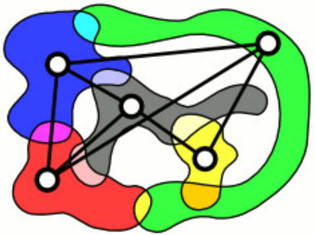

When H is non-planar:

- H-minor-free $\approx$ generalization of planar graphs
- H-induced-minor-free $\approx$ generalization of string graphs?

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H-induced-minor-free graphs have balanced separators of size $|H|^{\mathcal{O}(1)} \cdot \sqrt{m}$.

- Generalizes separator theorems on string graphs: $\mathcal{O}(m^{3/4}\sqrt{\log m})$ by [Fox and Pach'10], $\mathcal{O}(\sqrt{m}\log m)$ by [Matousek '14], $\mathcal{O}(\sqrt{m})$ by [Lee'17]

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Generalizes to $n^{\mathcal{O}_H(n^{2/3})}$ time algorithms for various problems that

- are about finding sparse induced subgraphs and
- can be solved in time $2^{\mathcal{O}(k \log k)} n^{\mathcal{O}(1)}$ parameterized by treewidth k

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Set $t = |H|^{\mathcal{O}(1)} \cdot \sqrt{m}$: If we get a separator, we are done.

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- Solution: Subdivide H three times, get a bad model of subdivided graph, reroute to a good model of H

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- Furthermore, H is a tree and the proof also gives NP-hardness
- Solves two open problems of [Fellows, Kratochvil, Middendorf, & Pfeiffer '95], who asked the existence of such H that is (1) planar (2) a tree

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3-Coloring $\leq$ Generalized 3-Coloring
$\leq$ Multicolored Induced 6-Disjoint Paths
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  - We use binary De Bruijn graphs

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4. Related: Complexity of k-disjoint induced paths on H-induced-minor-free graphs?

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2. Are all H-induced-minor-free graphs intersection graphs of connected subgraphs of H'-minor-free graphs?
3. Complexity of H-induced-minor-containment when minimal models of H are sparse? NP-hard? Quasipolynomial?
4. Related: Complexity of k-disjoint induced paths on H-induced-minor-free graphs? (or unit disk graphs?)

Conclusion

1. $|H|^{\mathcal{O}(1)} \cdot \sqrt{m}$ separator theorem on H-induced-minor-free graphs
2. $2^{o(n/\log^3 n)}$ hardness of H-induced-minor-containment for fixed tree H assuming ETH

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Thanks!