Linear-Time Algorithms for k-Edge-Connected Components, k-Lean Tree Decompositions, and More

Tuukka Korhonen

to appear in STOC'25

MPII seminar

4 March 2025

Graph connectivity problems

Near-linear-time and almost-linear-time algorithms for many problems...

- Edge connectivity (global edge min-cut), $\mathcal{O}(m \log^3 n)$ [Karger '96]
- Vertex connectivity (global vertex min-cut), $\mathcal{O}(m^{1+o(1)})$ [Li, Nanongkai, Panigrahi, Saranurak & Yingchareonthawornchai '21]
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k-Lean Tree Decompositions and More

Main technical result:

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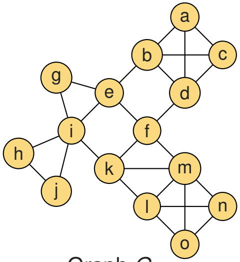
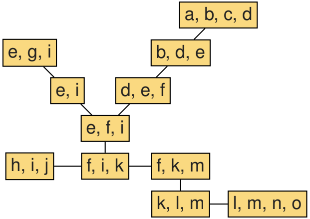
  - Previously $k^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ [Cygan, Komosa, Lokshtanov, Pilipczuk, Pilipczuk, Saurabh, Wahlström '21]
  - and $k^{\mathcal{O}(k)} m^{1+o(1)}$ [Anand, Lee, Li, Long, Saranurak '24] (suboptimal unbreakability parameters)

k-Lean Tree Decompositions



Graph G



A 3-lean tree decomposition of G

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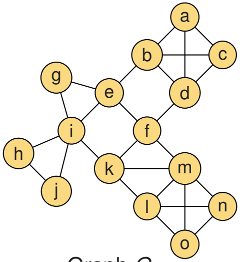

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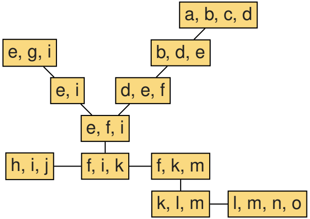

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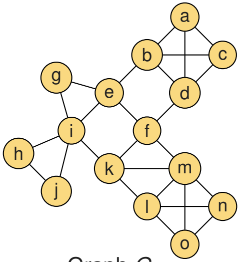

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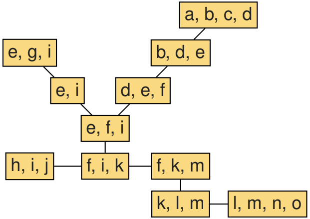

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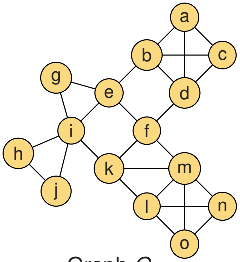

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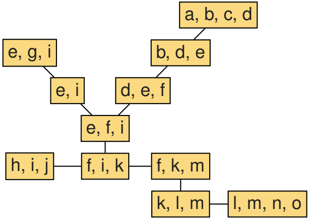

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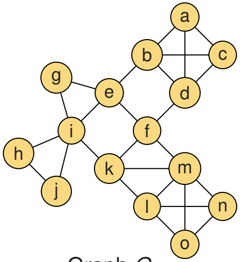

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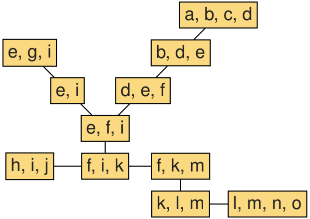

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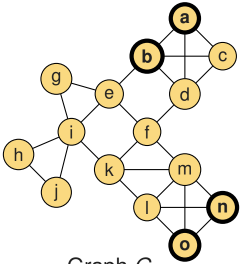

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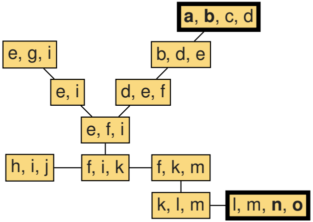

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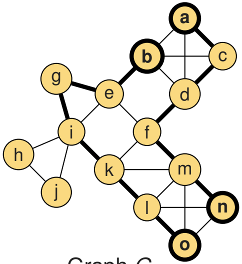

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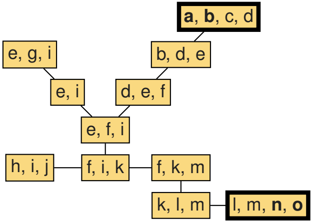

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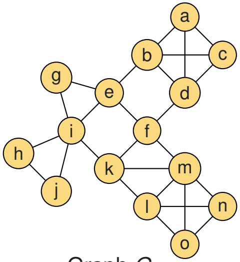

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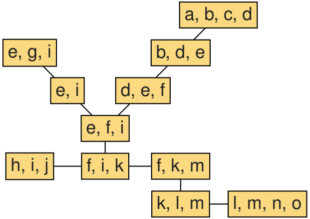

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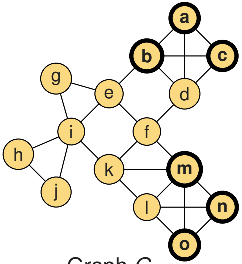

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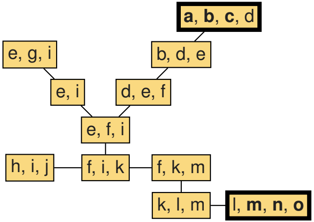

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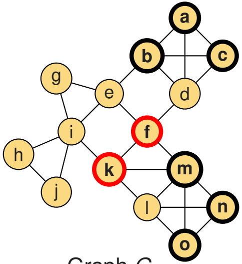
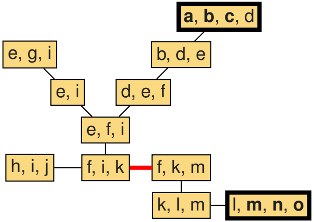
k-Lean Tree Decompositions



Graph G



A 3-lean tree decomposition of G

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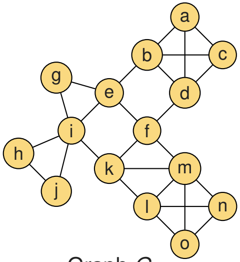

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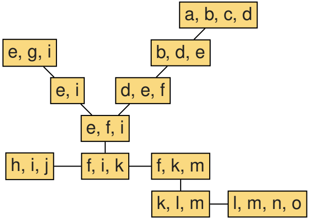

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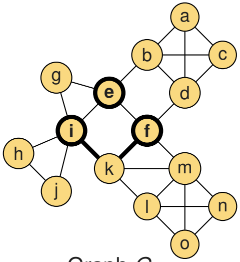

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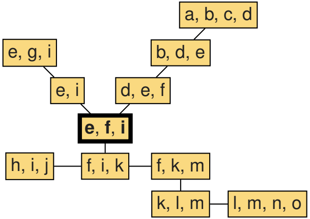

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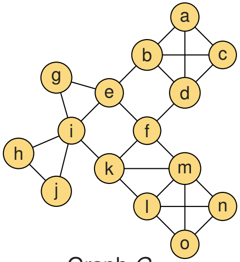

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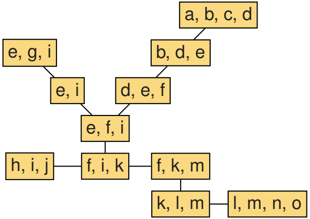

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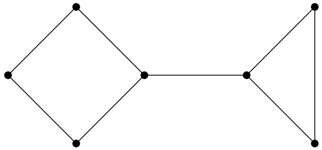
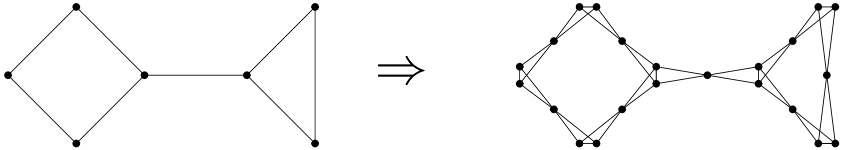
- Defined by [Thomas '90] (for $k = \infty$), and [Carmesin, Diestel, Hamann, and Hundertmark '14]

Reducing k-edge-connected components to k-lean tree decomposition



Reducing k-edge-connected components to k-lean tree decomposition

- Replace vertices by cliques of size k
- Create vertex for each edge and connect to the cliques corresponding to its endpoints



Reducing k-edge-connected components to k-lean tree decomposition

- Replace vertices by cliques of size k
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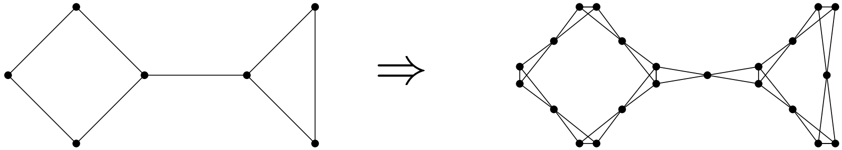
- Resulting k-lean tree decomposition gives a k-Gomory-Hu tree



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Input: A "weakly-k-lean" tree decomposition:

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Lemma

If there is improver algorithm with running time $f(k) \cdot m$, then there is an algorithm that in time $k^{\mathcal{O}(1)} \cdot f(k) \cdot m$ computes a k-lean tree decomposition.

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Case 2: No matching of size $\Omega(n) \Rightarrow$ manage to recurse in some other way...

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- Key tool: Decomposition by doubly well-linked separations

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