Finding Optimal Triangulations Parameterized by Edge Clique Cover

Tuukka Korhonen

Department of Computer Science, University of Helsinki, Finland

IPEC 2020
Online
December 15, 2020

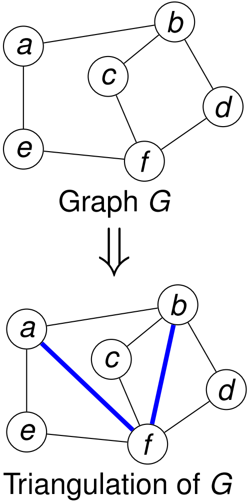
In this work

1. We argue that edge clique cover is well-motivated parameter for optimal triangulation (chordal completion) problems
   - e.g. treewidth and minimum fill-in

2. We use potential maximal cliques (PMCs) to give single exponential FPT algorithms parameterized by edge clique cover

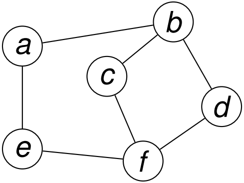
Optimal Triangulation Problems

- Given a graph, add edges to obtain a chordal graph minimizing some cost function

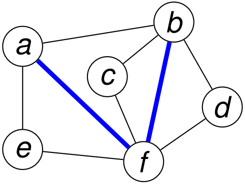


Optimal Triangulation Problems

- Given a graph, add edges to obtain a chordal graph minimizing some cost function
- Treewidth
  - Minimize the size of maximum clique



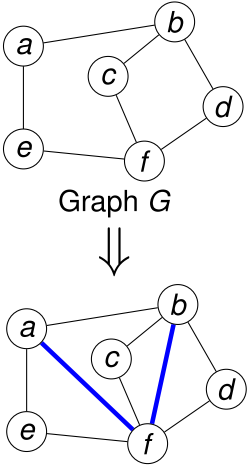
Graph G



Triangulation of G

Optimal Triangulation Problems

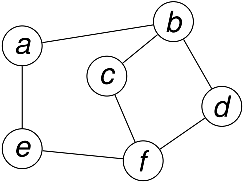
- Given a graph, add edges to obtain a chordal graph minimizing some cost function
- Treewidth
  - Minimize the size of maximum clique
- Minimum fill-in
  - Minimize the number of edges



Triangulation of G

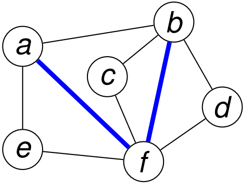
Optimal Triangulation Problems

- Given a graph, add edges to obtain a chordal graph minimizing some cost function
- Treewidth
  - Minimize the size of maximum clique
- Minimum fill-in
  - Minimize the number of edges
- Also fractional hypertreewidth



Graph G

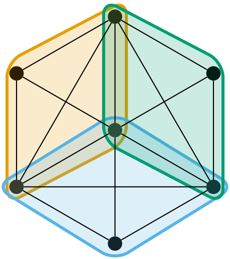
$\Downarrow$



Triangulation of G

Edge Clique Cover

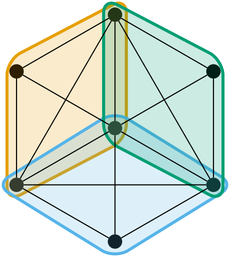
- Edge clique cover = set of cliques containing each edge
- Example: A graph with an edge clique cover of 3 cliques



Edge Clique Cover

- Edge clique cover = set of cliques containing each edge
- Example: A graph with an edge clique cover of 3 cliques

$\mathtt{cc} = 3$

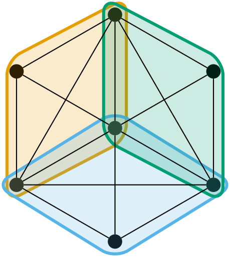


- We consider parameters:
  - size of minimum edge clique cover ($\mathtt{cc}$)
  - size of an edge clique cover given as input ($\mathtt{cc'}$)

Edge Clique Cover

- Edge clique cover = set of cliques containing each edge
- Example: A graph with an edge clique cover of 3 cliques

$\mathbf{cc} = 3$



- We consider parameters:
  - size of minimum edge clique cover ($\mathbf{cc}$)
  - size of an edge clique cover given as input ($\mathbf{cc'}$)
  - Note: Assuming ETH, no $2^{2^{o(\mathbf{cc})}}$ algorithm for computing $\mathbf{cc}$ [Cygan, Pilipczuk, and Pilipczuk, 2016]

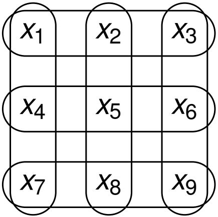
Plan

1. Motivation for edge clique cover
2. Potential maximal clique (PMC) framework
3. Our results
   - $O^*(3^{\mathbf{cc}})$ and $O^*(2^{\mathbf{cc}'})$ algorithms for optimal triangulation problems
4. Some proof sketches

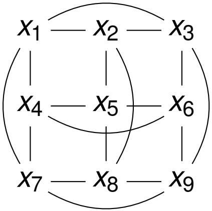
Motivation: CSP

- Treewidth and fractional hypertreewidth are central structural parameters of constraint satisfaction problems (CSPs)
  - Defined via triangulations of primal graph
  - which has $\texttt{cc} \leq m$, where m is the number of constraints

$n = 9$
$m = 6$



CSP/hypergraph



$n = 9$
$\texttt{cc} \leq 6$

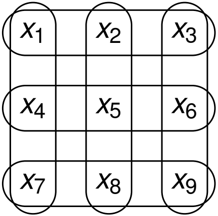
Primal graph

Motivation: CSP

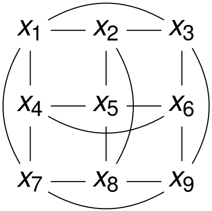
- Treewidth and fractional hypertreewidth are central structural parameters of constraint satisfaction problems (CSPs)
  - Defined via triangulations of primal graph
  - which has **cc** $\leq m$, where m is the number of constraints

- 708 of the 3072 hypergraphs in the Hyperbench library have $m \leq n/2$ [Fischl, Gottlob, Longo, and Pichler, 2019]

$n = 9$
$m = 6$



CSP/hypergraph



Primal graph

$n = 9$
cc ≤ 6

Motivation: Perfect phylogeny and Bayesian networks

Other similar constructions

1. Perfect phylogeny and its optimization variant
   - Reduction to weighted minimum fill-in [Gysel, 2014]
   - Graph with $\texttt{cc} \leq t$, where t is the number of taxa (species)

Motivation: Perfect phylogeny and Bayesian networks

Other similar constructions

1. Perfect phylogeny and its optimization variant
   - Reduction to weighted minimum fill-in [Gysel, 2014]
   - Graph with $\mathtt{cc} \leq t$, where t is the number of taxa (species)
   - Real instances:
     - Mammal mitochondrial sequences: $n = 245$, $\mathtt{cc} \leq 7$
     - Indo-European languages: $n = 864$, $\mathtt{cc} \leq 24$

Motivation: Perfect phylogeny and Bayesian networks

Other similar constructions

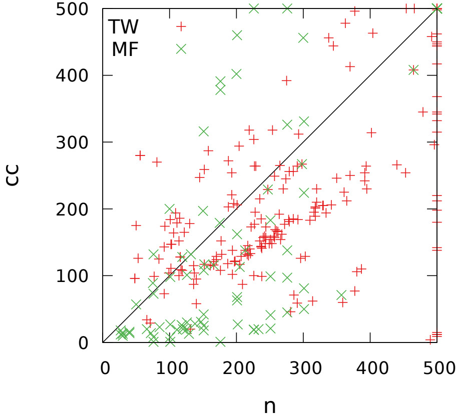
1. Perfect phylogeny and its optimization variant
   - Reduction to weighted minimum fill-in [Gysel, 2014]
   - Graph with $\texttt{cc} \leq t$, where t is the number of taxa (species)
   - Real instances:
     - Mammal mitochondrial sequences: $n = 245$, $\texttt{cc} \leq 7$
     - Indo-European languages: $n = 864$, $\texttt{cc} \leq 24$

2. Bayesian networks
   - Treewidth of moral graph important
   - Moral graph has $\texttt{cc} \leq n'$, where n' is the number of non-root nodes

Motivation: Explaining success of PMCs

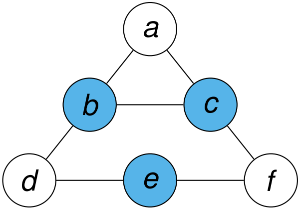
- PACE 2017: Top 4 solvers on minimum fill-in track and rank 2 solver on treewidth track based on potential maximal cliques

- Implementations based on PMCs:
  - Treewidth [Tamaki, 2019]
  - Fractional hypertreewidth [Korhonen, Berg, and Järvisalo, 2019]
  - Phylogenetics [Korhonen and Järvisalo, 2020]
  - Enumeration of minimal triangulations [Ravid, Medini, and Kimelfeld, 2019]

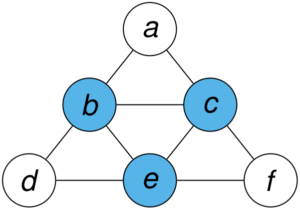


PMC Framework

Potential maximal clique (PMC) = vertex set that is a maximal clique in some minimal triangulation [Bouchitté and Todinca, 2001]



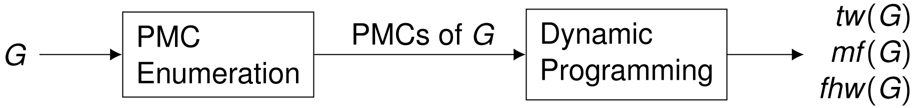
Graph G with PMC $\{b, c, e\}$



A minimal triangulation of G

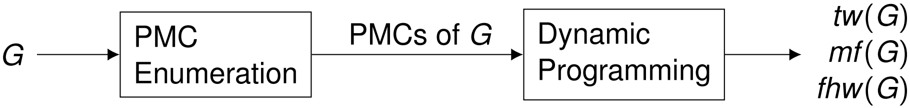
PMC Framework

Potential maximal clique (PMC) = vertex set that is a maximal clique in some minimal triangulation [Bouchitté and Todinca, 2001]



PMC Framework

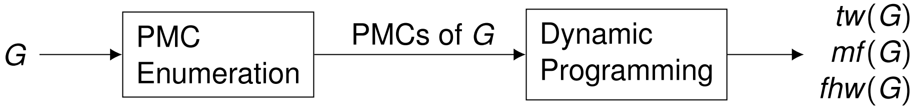
Potential maximal clique (PMC) = vertex set that is a maximal clique in some minimal triangulation [Bouchitté and Todinca, 2001]



- $O^*(\#PMCs)$ [Fomin, Kratsch, Todinca, and Villanger, 2008]

PMC Framework

Potential maximal clique (PMC) = vertex set that is a maximal clique in some minimal triangulation [Bouchitté and Todinca, 2001]

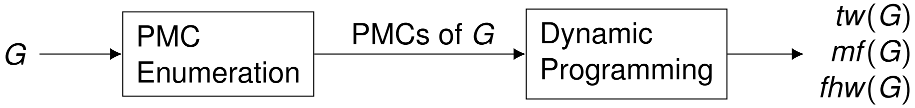


- $O^*(\#minseps^2)$ [Bouchitté and Todinca, 2002]
- $O(1.74^n)$ [Fomin, Todinca, and Villanger, 2015]
- $O^*(4^{vc})$ [Fomin, Liedloff, Montealegre, and Todinca, 2018]

- $O^*(\#PMCs)$ [Fomin, Kratsch, Todinca, and Villanger, 2008]

PMC Framework

Potential maximal clique (PMC) = vertex set that is a maximal clique in some minimal triangulation [Bouchitté and Todinca, 2001]



- $O^*(\#minseps^2)$ [Bouchitté and Todinca, 2002]
- $O(1.74^n)$ [Fomin, Todinca, and Villanger, 2015]
- $O^*(4^{vc})$ [Fomin, Liedloff, Montealegre, and Todinca, 2018]
- $O^*(3^{\mathbf{cc}})$ [This work]

- $O^*(\#PMCs)$ [Fomin, Kratsch, Todinca, and Villanger, 2008]

Main Contributions

Parameter $\mathbf{cc}$ – size of minimum edge clique cover

Theorems

- A graph has at most $2^{\mathbf{cc}}$ minimal separators and $3^{\mathbf{cc}}$ PMCs
- PMCs can be enumerated in time $O^*(3^{\mathbf{cc}})$
- There are graphs with $\Omega(2^{\mathbf{cc}})$ minseps and $\Omega(3^{\mathbf{cc}})$ PMCs

Main Contributions

Parameter $\mathbf{cc}$ – size of minimum edge clique cover

Theorems

- A graph has at most $2^{\mathbf{cc}}$ minimal separators and $3^{\mathbf{cc}}$ PMCs
- PMCs can be enumerated in time $O^*(3^{\mathbf{cc}})$
- There are graphs with $\Omega(2^{\mathbf{cc}})$ minseps and $\Omega(3^{\mathbf{cc}})$ PMCs

Corollary

- All problems that can be solved with PMC DP in time $O^*(\#PMCs)$ can be solved in time $O^*(3^{\mathbf{cc}})$.
- Including treewidth, weighted minimum fill-in, fractional hypertreewidth, feedback vertex set...

Further Contributions

Parameter $\mathtt{cc'}$ – size of an edge clique cover given as input

Theorem

Treewidth, minimum fill-in, and chordal sandwich can be solved in $O^*(2^{\mathtt{cc'}})$ time

Further Contributions

Parameter $\texttt{cc'}$ – size of an edge clique cover given as input

Theorem

Treewidth, minimum fill-in, and chordal sandwich can be solved in $O^*(2^{\texttt{cc'}})$ time

Corollary

Perfect phylogeny can be solved in time $O^*(2^t)$, where t is the number of taxa

Further Contributions

Parameter $\mathtt{cc}'$ – size of an edge clique cover given as input

Theorem

Treewidth, minimum fill-in, and chordal sandwich can be solved in $O^*(2^{\mathtt{cc}'})$ time

Corollary

Perfect phylogeny can be solved in time $O^*(2^t)$, where t is the number of taxa

Theorem

Polynomial space algorithms with time complexities $O^*(9^{\mathtt{cc}'})$ and $O^*(9^{\mathtt{cc}+O(\log^2 \mathtt{cc})})$ for problems in PMC framework.

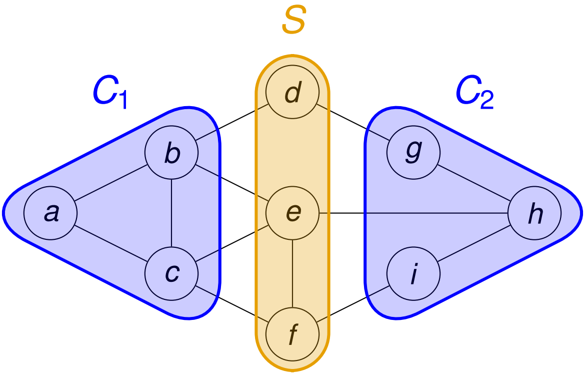
Proofs

I will sketch the proof of the main result

1. Preliminaries: Minimal separators and blocks
2. Lemma: Each block corresponds to a unique subset of edge clique cover
3. Theorem: $O^*(3^{\mathbf{cc}})$ PMC enumeration algorithm

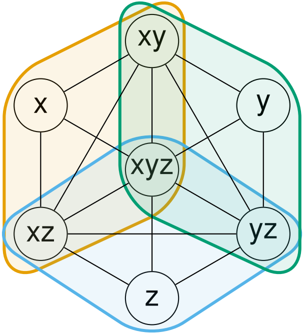
Preliminaries: Minimal Separators and Blocks

- A connected vertex set C is a block if its neighborhood $N(C) = S$ is a minimal separator
- Each minimal separator has at least 2 associated blocks



The Lemma

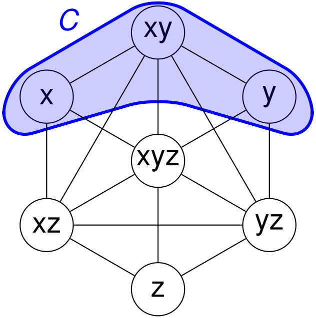
- Let $\mathcal{W}$ be an edge clique cover
- For vertex v, let $\mathcal{W}[v] \subseteq \mathcal{W}$ be the cliques that contain v



$\mathcal{W} = \{x, y, z\}$

The Lemma

- Let $\mathcal{W}$ be an edge clique cover
- For vertex v, let $\mathcal{W}[v] \subseteq \mathcal{W}$ be the cliques that contain v
- For block C, let $\mathcal{W}[C] = \bigcup_{v \in C} \mathcal{W}[v]$



$$\mathcal{W} = \{x, y, z\}$$

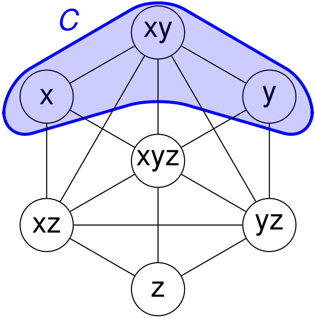
$$\mathcal{W}[C] = \{x, y\}$$

The Lemma

- Let $\mathcal{W}$ be an edge clique cover
- For vertex v, let $\mathcal{W}[v] \subseteq \mathcal{W}$ be the cliques that contain v
- For block C, let $\mathcal{W}[C] = \bigcup_{v \in C} \mathcal{W}[v]$

Lemma

$v \in C$ if and only if $\mathcal{W}[v] \subseteq \mathcal{W}[C]$



$$\mathcal{W} = \{x, y, z\}$$

$$\mathcal{W}[C] = \{x, y\}$$

The Lemma

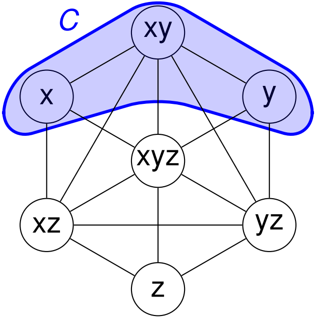
- Let $\mathcal{W}$ be an edge clique cover
- For vertex v, let $\mathcal{W}[v] \subseteq \mathcal{W}$ be the cliques that contain v
- For block C, let $\mathcal{W}[C] = \bigcup_{v \in C} \mathcal{W}[v]$

Lemma

$v \in C$ if and only if $\mathcal{W}[v] \subseteq \mathcal{W}[C]$

Proof idea

- Only if from definition
- Vertex v with $\mathcal{W}[v] \subseteq \mathcal{W}[C]$ must be in $C \cup N(C)$
- However, $v \notin N(C)$ by properties of minimal separators



$\mathcal{W} = \{x, y, z\}$

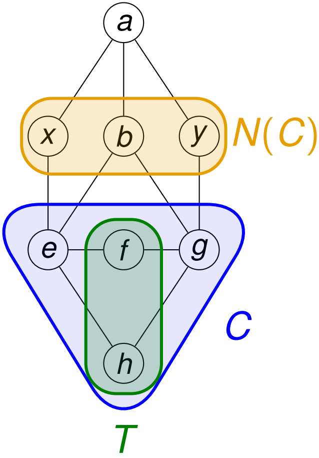
$\mathcal{W}[C] = \{x, y\}$

PMC enumeration

- Bouchitté–Todinca enumeration [Bouchitté and Todinca, 2002]
  - Multiple "optimal" steps
  - Single bottleneck step

Algorithm: PMC-ENUM bottleneck

foreach *Block C* **do**
    foreach *Non-adjacent* $\{x, y\} \subseteq N(C)$ **do**
        foreach *Minsep T of* $G[C \cup \{x, y\}]$ **do**
            if *IsPMC*$(N(C) \cup T)$ **then**
                Output$(N(C) \cup T)$

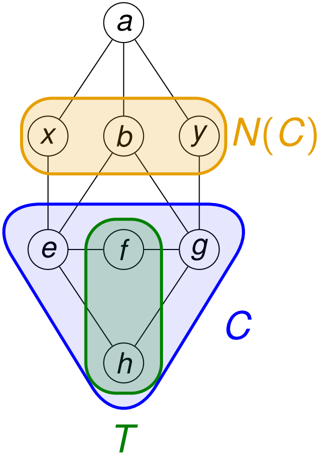


PMC enumeration

- Bouchitté–Todinca enumeration [Bouchitté and Todinca, 2002]
  - Multiple "optimal" steps
  - Single bottleneck step

Algorithm: PMC-ENUM bottleneck

foreach *Block C* **do**
  foreach *Non-adjacent* $\{x, y\} \subseteq N(C)$ **do**
    foreach *Minsep T of* $G[C \cup \{x, y\}]$ **do**
      if *IsPMC*$(N(C) \cup T)$ **then**
        Output$(N(C) \cup T)$



- $\mathcal{W}[C]$ is ECC of $G[C \cup \{x, y\}]$
- $\implies$ Inner loop complexity $O^*(2^{|\mathcal{W}[C]|})$
- In total $\sum_C O^*(2^{|\mathcal{W}[C]|}) = O^*(3^{\mathbf{cc}})$

Ideas of Other Results

- $O^*(2^{\mathbf{cc}'})$ algorithms
  - Same DP states as in PMC framework, but instead of using PMCs for transitions use fast subset convolution [Björklund, Husfeldt, Kaski, and Koivisto, 2007]

- Polynomial space $O^*(9^{\mathbf{cc}'})$ and $O^*(9^{\mathbf{cc}+O(\log^2 \mathbf{cc})})$ algorithms
  - Branch from PMCs that are balanced separators with respect to edge clique cover

Conclusion

- Edge clique cover is a natural parameter for optimal triangulation problems in multiple settings
- We used PMC framework to give $O^*(3^{\mathbf{cc}})$ and $O^*(2^{\mathbf{cc}'})$ time algorithms for such problems
- Our combinatorial bounds are tight, but unknown if the algorithms for individual problems are optimal
- It remains as an open problem to give an $O^*(\#PMCs)$ time PMC enumeration algorithm

The end

Thank you for your attention!

Bibliography

Andreas Björklund, Thore Husfeldt, Petteri Kaski, and Mikko Koivisto. Fourier meets Möbius: fast subset convolution. In *Proceedings of the 39th Annual ACM Symposium on Theory of Computing*, pages 67–74. ACM, 2007. doi: 10.1145/1250790.1250801.

Vincent Bouchitté and Ioan Todinca. Treewidth and minimum fill-in: Grouping the minimal separators. *SIAM Journal on Computing*, 31(1):212–232, 2001. doi: 10.1137/S0097539799359683.

Vincent Bouchitté and Ioan Todinca. Listing all potential maximal cliques of a graph. *Theoretical Computer Science*, 276(1-2): 17–32, 2002. doi: 10.1016/S0304-3975(01)00007-X.

Marek Cygan, Marcin Pilipczuk, and Michal Pilipczuk. Known algorithms for edge clique cover are probably optimal. *SIAM Journal on Computing*, 45(1):67–83, 2016. doi: 10.1137/130947076.

Wolfgang Fischl, Georg Gottlob, Davide Mario Longo, and Reinhard Pichler. HyperBench: A benchmark and tool for hypergraphs and empirical findings. In *Proceedings of the 38th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems*, pages 464–480, 2019. doi: 10.1145/3294052.3319683.

Fedor V. Fomin, Dieter Kratsch, Ioan Todinca, and Yngve Villanger. Exact algorithms for treewidth and minimum fill-in. *SIAM Journal on Computing*, 38(3):1058–1079, 2008. doi: 10.1137/050643350.

Fedor V. Fomin, Ioan Todinca, and Yngve Villanger. Large induced subgraphs via triangulations and CMSO. *SIAM Journal on Computing*, 44(1):54–87, 2015. doi: 10.1137/140964801.

Fedor V. Fomin, Mathieu Liedloff, Pedro Montealegre, and Ioan Todinca. Algorithms parameterized by vertex cover and modular width, through potential maximal cliques. *Algorithmica*, 80(4):1146–1169, 2018. doi: 10.1007/s00453-017-0297-1.

Rob Gysel. Minimal triangulation algorithms for perfect phylogeny problems. In *Language and Automata Theory and Applications - 8th International Conference*, volume 8370 of *LNCS*, pages 421–432. Springer, 2014. doi: 10.1007/978-3-319-04921-2_34.

Tuukka Korhonen and Matti Järvisalo. Finding most compatible phylogenetic trees over multi-state characters. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence*, pages 1544–1551. AAAI Press, 2020. doi: 10.1609/aaai.v34i02.5514.

Tuukka Korhonen, Jeremias Berg, and Matti Järvisalo. Solving graph problems via potential maximal cliques: An experimental evaluation of the Bouchitté-Todinca algorithm. *ACM Journal of Experimental Algorithmics*, 24(1):1.9:1–1.9:19, 2019. doi: 10.1145/3301297.

Noam Ravid, Dori Medini, and Benny Kimelfeld. Ranked enumeration of minimal triangulations. In *Proceedings of the 38th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems*, pages 74–88. ACM, 2019. doi: 10.1145/3294052.3319678.

Hisao Tamaki. Positive-instance driven dynamic programming for treewidth. *Journal of Combinatorial Optimization*, 37(4): 1283–1311, 2019. doi: 10.1007/s10878-018-0353-z.

Bonus: Exact bounds

- $S(n, k)$ – number of ways to partition n-element set into k non-empty parts
- There are at most $S(\texttt{cc}, 2)$ minseps
- There are at most $S(\texttt{cc}, 3) + \texttt{cc}$ PMCs