Computing Tree Decompositions with Small Independence Number

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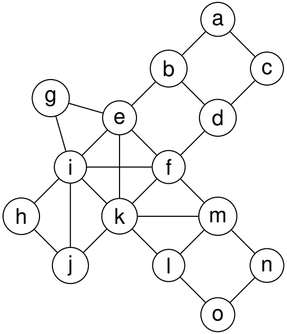
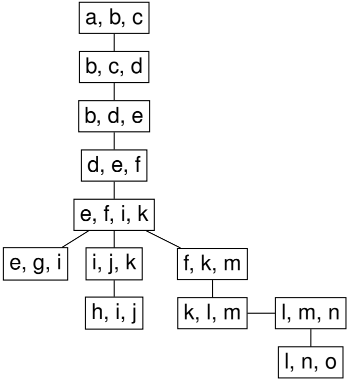
Tree Decompositions





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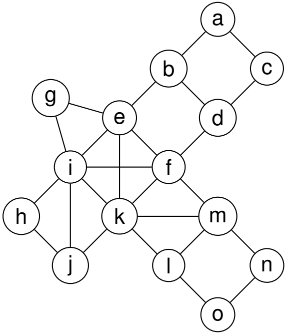

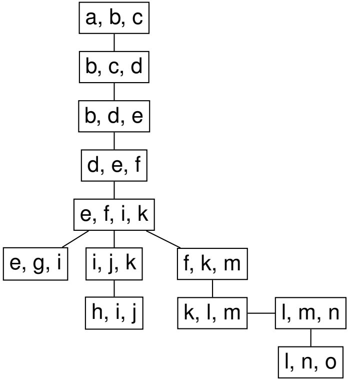

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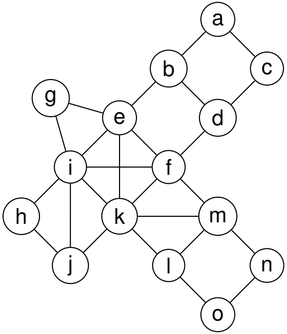

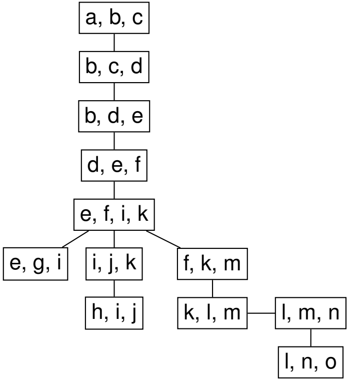

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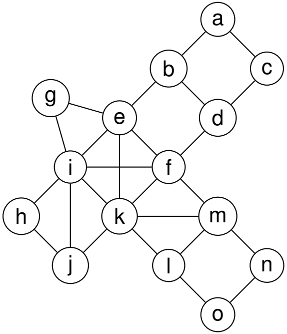

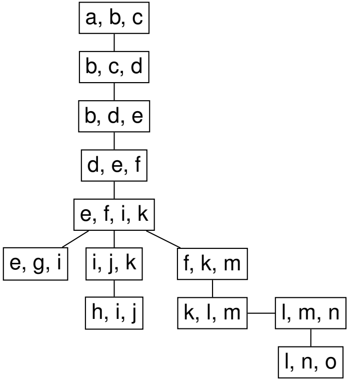

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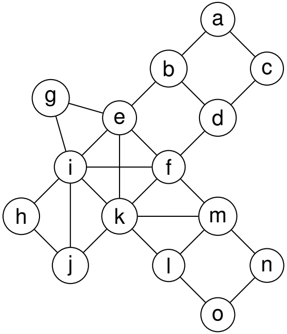

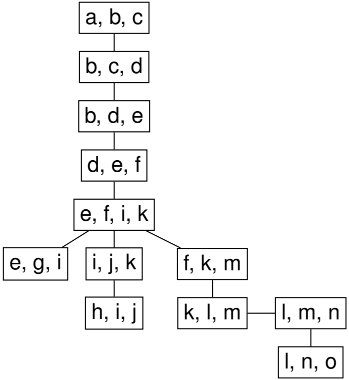

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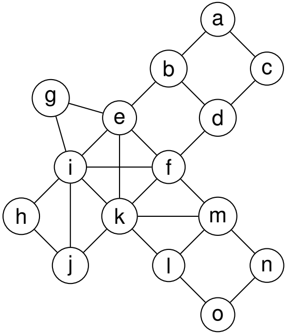

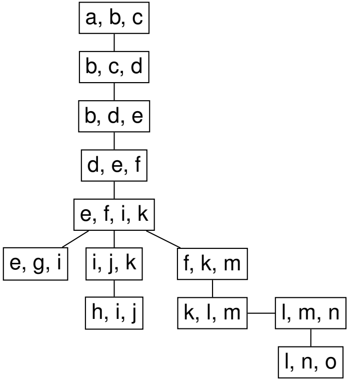

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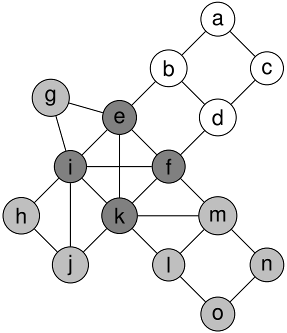
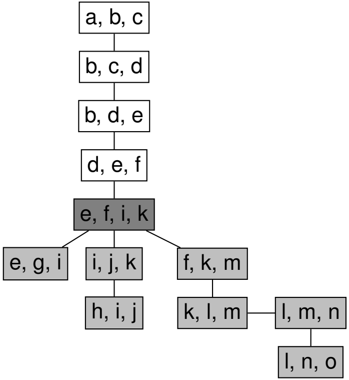
5. Treewidth = minimum width of a tree decomposition

Dynamic programming for maximum independent set





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$\mathrm{dp}[t][S]$ = maximum independent set I below t with $I \cap B_t = S$

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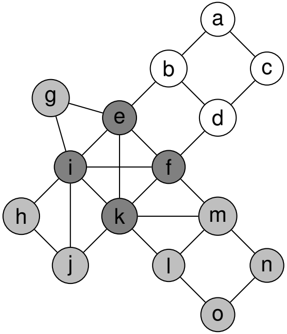

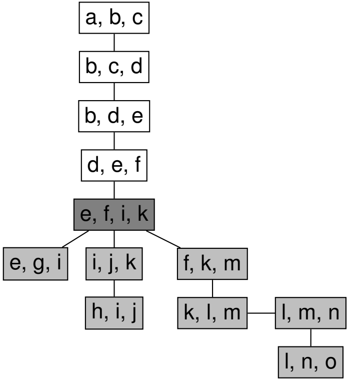

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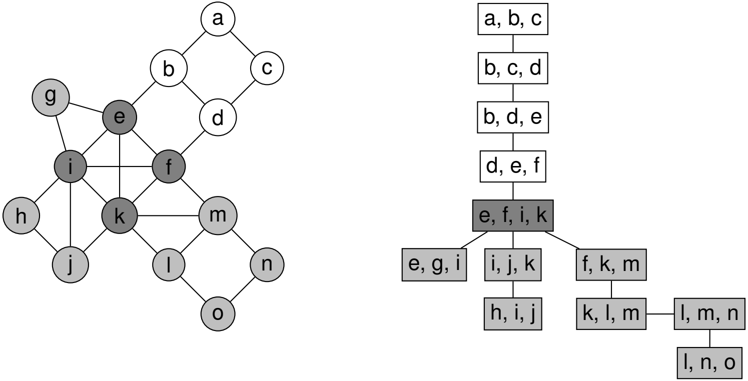

For every node t and **independent** subset $S \subseteq B_t$

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$\#IS(B_t)$ states per node

When can we bound $\#IS(B_t)$?

What kind of tree decompositions have bounded $\#IS(B_t)$?

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Tree-independence number: tree-$\alpha(G) = \min_{TD} \alpha(TD)$

- Introduced by [Yolov, SODA'18] and independently by [Dallard, Milanič, & Storgel, '21]

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Important subroutine: Computing the tree decomposition!

Our results

Theorem

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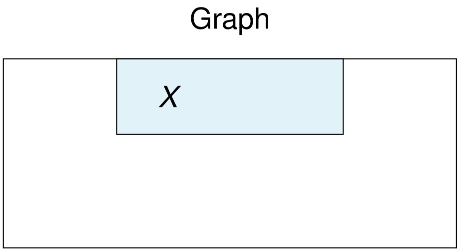
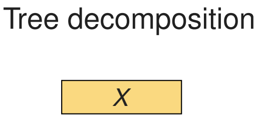
The Algorithm

Recursive top-down construction in Robertson-Seymour fashion





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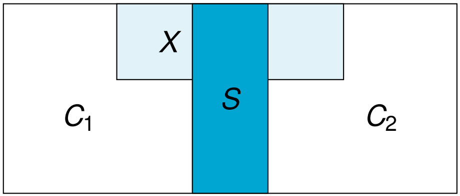

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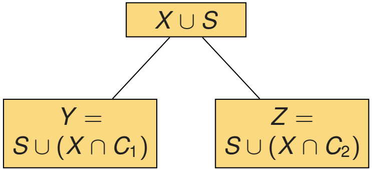

Balanced separator S with components C_1 and C_2

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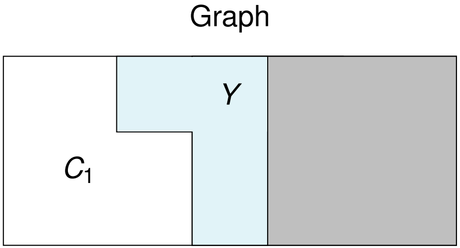

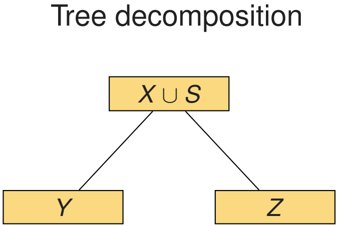

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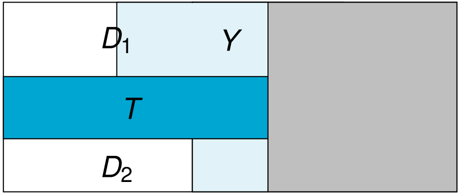

Balanced separator T with components D_1 and D_2

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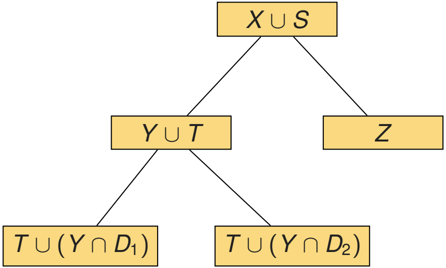

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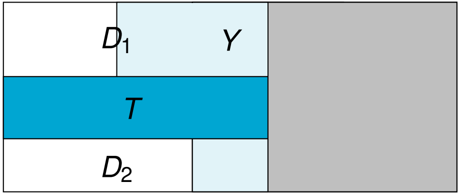

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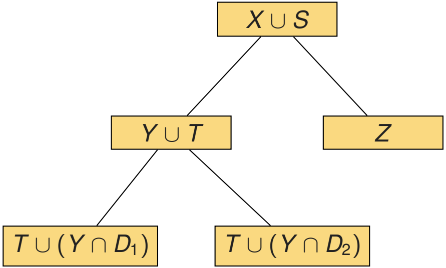

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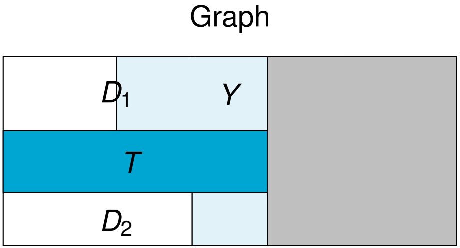

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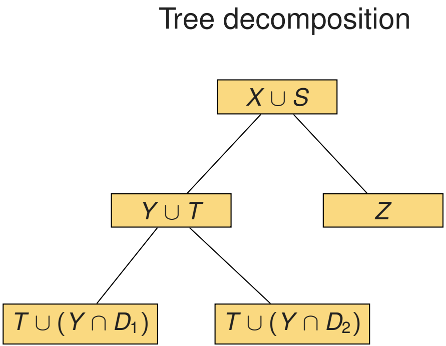

Theorem (Informal)

If for any vertex set X with $\alpha(X) = 9k$ we can find a separation (C_1, S, C_2) so that $\alpha(S) \leq 2k$, $\alpha(X \cap C_1) \leq 7k$, and $\alpha(X \cap C_2) \leq 7k$, then we get 11-approximation

Balanced separators

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   3.2 Branching + linear programming to find the separator

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  - More general tree decomposition based parameters for which independent set and related problems are XP?

Conclusion

- $2^{\mathcal{O}(k^2)} n^{\mathcal{O}(k)}$ time 8-approximation algorithm for tree-independence number
- para-NP-hardness of computing tree-independence number exactly
- Open problems:
  - Complexity of deciding tree-$\alpha(G) \leq k$ for $k = 2, 3$?
  - Is deciding tree-$\alpha(G) \leq k$ for unbounded k in NP or $\sum_2^P$-hard?
  - More general tree decomposition based parameters for which independent set and related problems are XP?

Thank you!