Fixed-Parameter Tractability of Maximum Colored Path and Beyond

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UNIVERSITY OF BERGEN

joint work with
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Helsinki CS Theory Seminar

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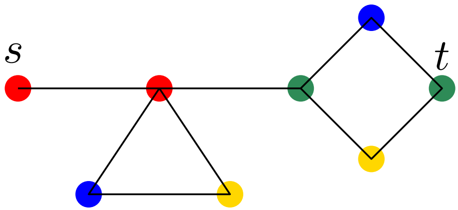
Maximum Colored Path

MAXIMUM COLORED *s*, *t*-PATH

Input:	Vertex-colored undirected graph, vertices *s* and *t*, and an integer *k*.
Task:	Find an *s*, *t*-path containing at least *k* different colors.



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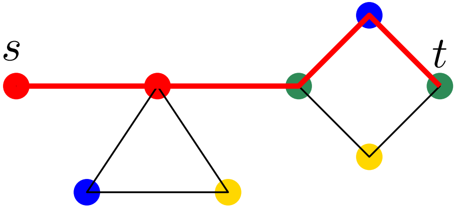

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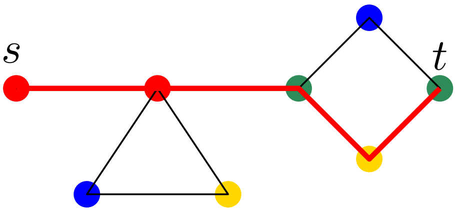

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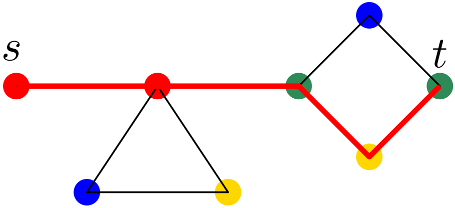

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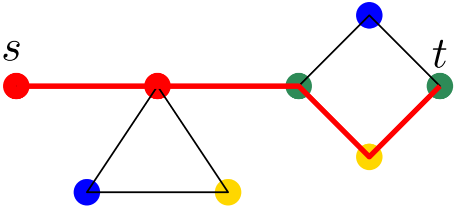

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- A path does not contain repeated vertices
- A color may repeat multiple times in the path, and it can contain more than k colors

Fixed-parameter tractability of Maximum Colored Path

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Input: Vertex-colored undirected graph, vertices s and t, and an integer k.

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Theorem

There is a $2^k n^{\mathcal{O}(1)}$ time randomized algorithm for maximum colored s,t-path. Moreover, the algorithm returns the shortest solution.

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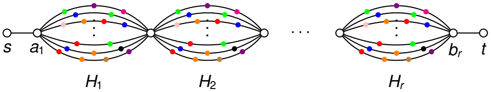

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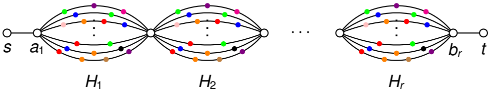

- NP-hard for directed graphs already when $k = 2$

Application: Longest path

LONGEST s, t-PATH

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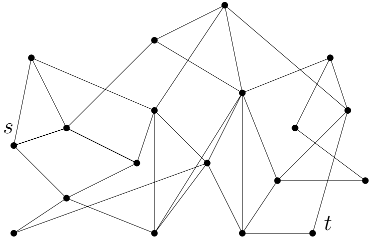

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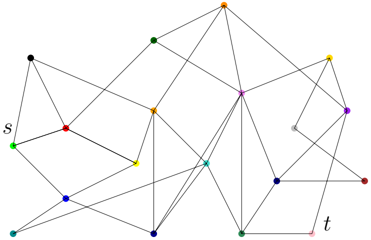

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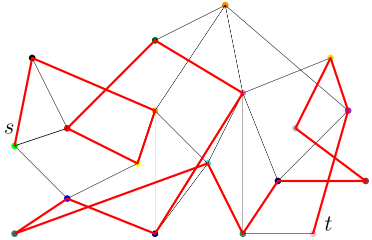

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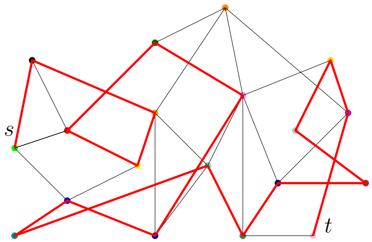

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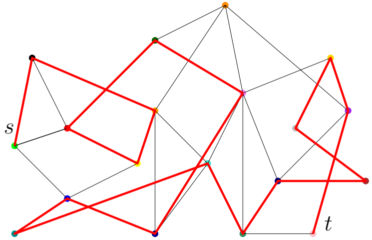

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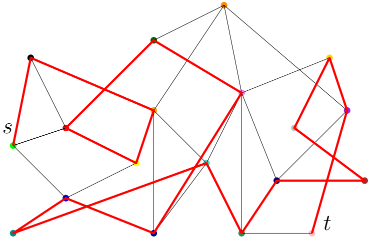

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- Previous best algorithm $4.884^k n^{\mathcal{O}(1)}$ time [Fomin, Lokshtanov, Panolan, Saurabh, Zehavi'18]
- In contrast, there is a $1.66^k n^{\mathcal{O}(1)}$ time algorithm for longest path [Björklund, Husfeldt, Kaski, Koivisto'17]

Application: T-cycle

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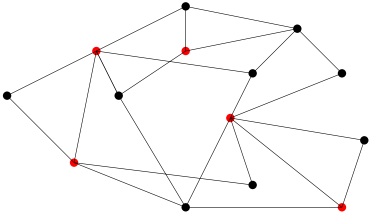

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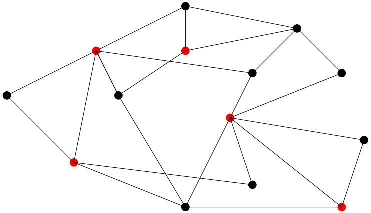

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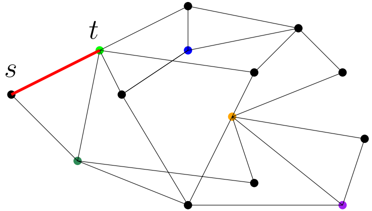

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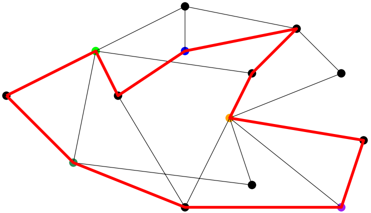

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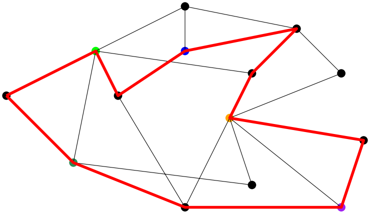

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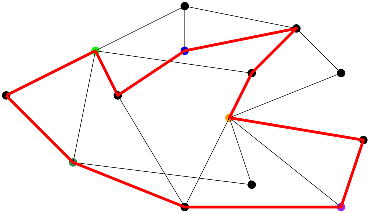

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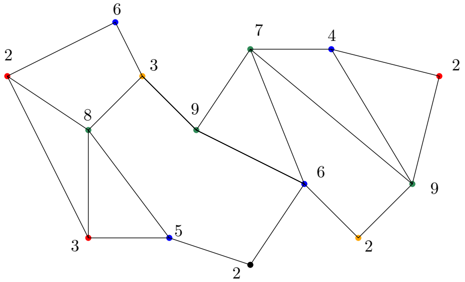
- Allows for generalizations, e.g., arbitrarily large T

Beyond Maximum Colored Path

Input:

- Colored positive-integer weighted undirected graph



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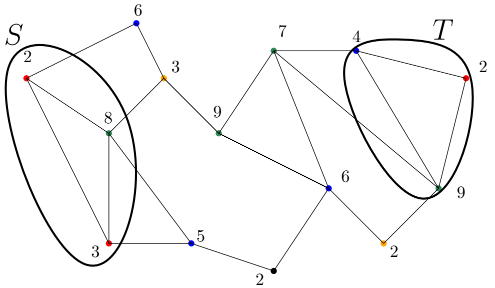

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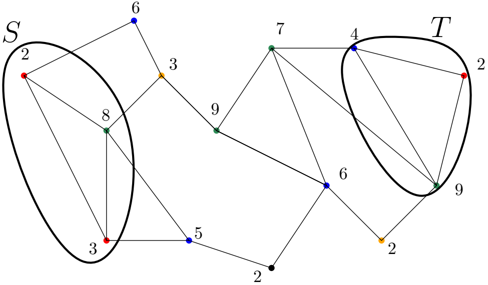

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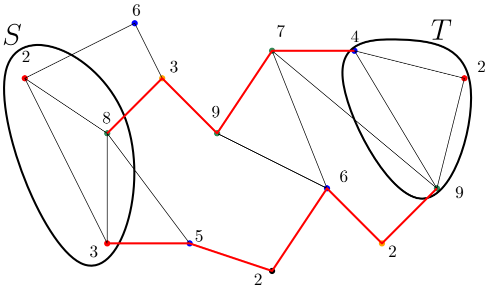

Here, $p = 2$

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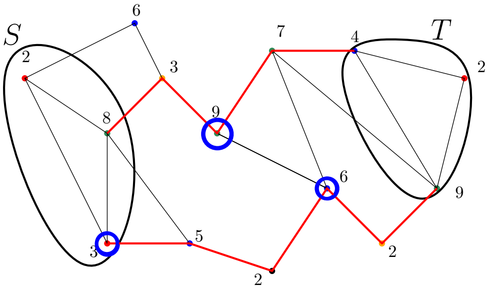

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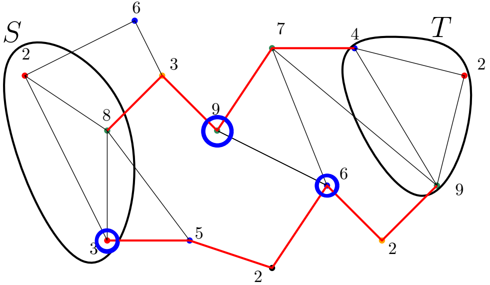

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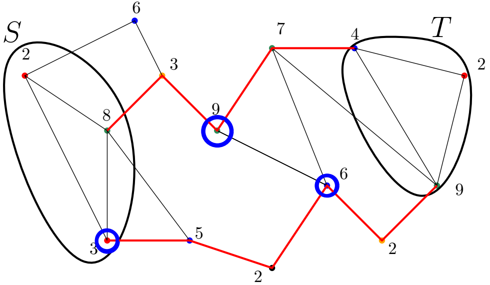

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Main theorem: Randomized algorithm with time complexity $2^{k+p}n^{\mathcal{O}(1)}w$.

More applications

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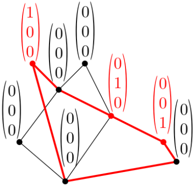
- Generalization from colored graphs to (graph, matroid) pairs represented over a finite field of order q with a $2^{\mathcal{O}(k^2 \log(q+p))}$ factor overhead



The algorithm

Outline

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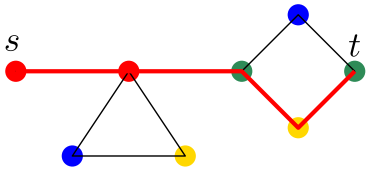

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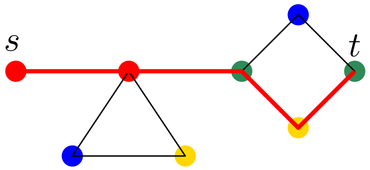

- Algorithm based on algebraic approach, extending ideas that were developed by [Björklund, Husfeldt, Taslaman'12], [Björklund'14], [Björklund, Husfeldt, Kaski, Koivisto'17] for T-cycle, hamiltonicity, and k-path

Algebraic approach

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- Characteristic 2?
  - $x + x = 0$ for any x

Design of the polynomial

Idea: Design a polynomial where monomials correspond to *labeled walks*

Design of the polynomial

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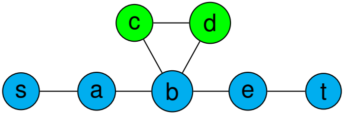
- Evaluating a polynomial over walks is easier than over paths
- Argue that monomials corresponding to non-path walks and paths with less than k colors cancel out (by $x + x = 0$), and only the contribution of k-colored paths remains

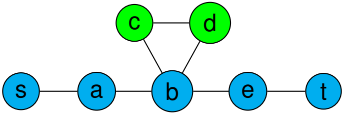
Labeled walks



$k = 2$

- (s, t)-walk of length ℓ is a sequence $s = v_1 \dots v_\ell = t$ of ℓ adjacent vertices

Labeled walks



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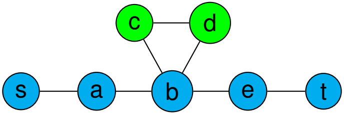

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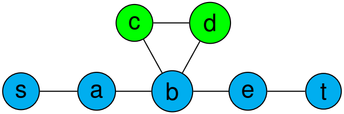

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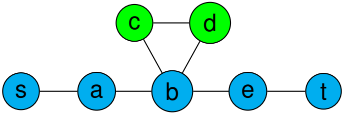

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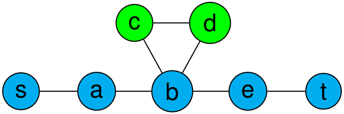

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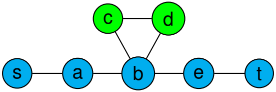
  - Intention of labels: Indicate k vertices with different colors
- A labeled (s, t)-walk is *labeled-digon-free* if it has no subwalks of form $a\overset{1}{b}a$

Polynomial over labeled walks



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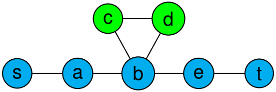

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Definition:

- For each edge uv associate variable $f_e(uv)$
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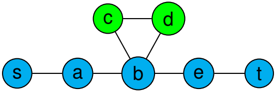

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For a labeled walk W, associate monomial $f(W)$ that is product of edge variables, vertex variables of labeled vertices, and color-label pair variables corresponding to labeled vertices

$$f(sa\overset{1}{b}c\overset{2}{d}bet) = f_e(sa)f_e(ab)f_e(bc)f_e(cd)f_e(db)f_e(be)f_e(et)f_v(b)f_v(d)f_c(\bullet, 1)f_c(\bullet, 2)$$

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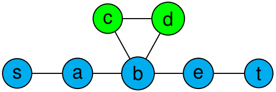

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For integer ℓ, let $\mathcal{C}_\ell$ be the family of labeled-digon-free labeled (s, t)-walks of length ℓ

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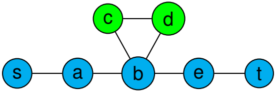

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For integer ℓ, let $\mathcal{C}_\ell$ be the family of labeled-digon-free labeled (s, t)-walks of length ℓ

- Define $f(\mathcal{C}_\ell) = \sum_{W \in \mathcal{C}_\ell} f(W)$

Algebraic approach

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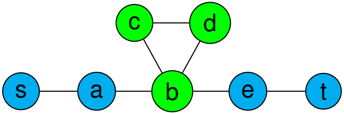
3. $f(\mathcal{C}_\ell)$ can be evaluated in time $2^k n^{\mathcal{O}(1)}$

Yes-instances



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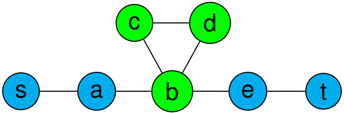

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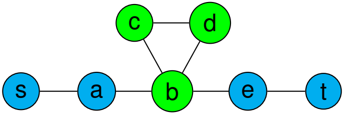

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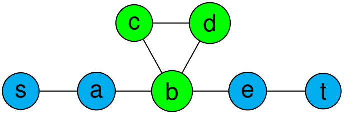

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$\Rightarrow$ $f(\mathcal{C}_5)$ is non-zero

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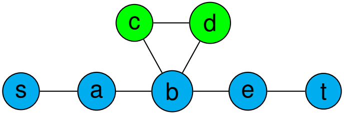
Three different cancellation arguments as building blocks for ϕ:

- Bijective-labeling based cancellation (from [Björklund'14], [Björklund, Husfeldt, Kaski, Koivisto'17])
- Cycle-reversal based cancellation (from [Björklund, Husfeldt, Taslaman'12])
- Label-swap based cancellation

Bijective-labeling based cancellation



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Consider labeled walks where two vertices of the same color are labeled

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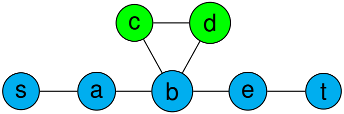

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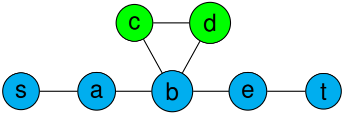

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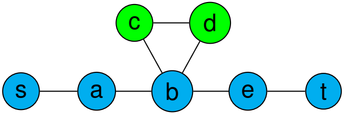

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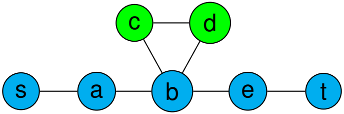

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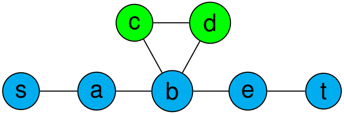
- Now we can work with family $\mathcal{C}_\ell^* \subseteq \mathcal{C}_\ell$ containing labeled walks where all labeled vertices have different colors

Cycle-reversal based cancellation



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- Labeled walk $s\overset{1}{a}b\overset{2}{c}dbet \in \mathcal{C}_7^*$ contributes monomial $f_e(sa)f_e(ab)f_e(bc)f_e(cd)f_e(db)f_e(be)f_e(et)f_v(a)f_v(c)f_c(\bullet, 1)f_c(\bullet, 2)$

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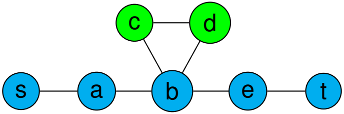

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- Labeled walk $\color{red}{s\overset{1}{a}b\overset{2}{c}dbet} \in \mathcal{C}_7^*$ contributes monomial $f_e(sa)f_e(ab)f_e(bc)f_e(cd)f_e(db)f_e(be)f_e(et)f_v(a)f_v(c)f_c(\color{blue}{\bullet}, 1)f_c(\color{green}{\bullet}, 2)$
- Reverse the cycle $\color{red}{b\overset{2}{c}db}$

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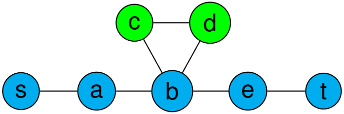

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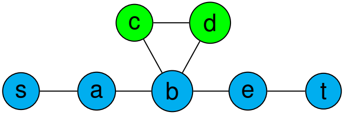

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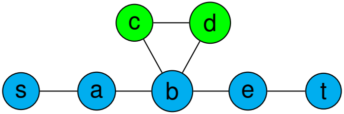

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- This does **not** cleanly handle things, need more arguments and case analysis

Label-swap based cancellation

- Consider a labeled walk $sabcd\overset{1}{a}bt$ (with only one label)

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- The label swap operation could also create labeled digons!

Cancellation arguments

- Bad news: Not clear when to apply cycle reversal and when label swap

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- Good news: We managed to construct ϕ, but it is very complicated

Definition 3 (The function ϕ). *Let $\mathcal{W} = (W^1, \dots, W^p)$ be a proper barren labeled walkage of order p. For each $i \in [p]$, denote $W^i = ((v^i_1, \dots, v^i_{\ell_i}), (r^i_1, \dots, r^i_{\ell_i}))$. The value $\phi(\mathcal{W})$ is defined, in some cases recursively, by selecting the first matching case from the following list:*

A. *if the vertex v^1_1 occurs only once in $\mathcal{W}$:*

   1. *if $\ell_1 \geq 2$, then $\phi(\mathcal{W}) = W^1[1,1] \diamond \phi(W^1[2,\ell_1], W^2, \dots, W^p)$.*
   2. *otherwise (i.e., $\ell_1 = 1$), $\phi(\mathcal{W}) = W^1 \sqcup \phi(W^2, \dots, W^p)$.*

B. *if the vertex v^1_1 occurs in at least three different walks W^i:*
   There must be at least two different walks W^i that contain v^1_1 but do not contain it as labeled. Let i, j be the two smallest indices so that both W^i and W^j contain v^1_1 but do not contain it as labeled. Let a be the index of the first occurrence of v^1_1 in W^i and b be the index of the first occurrence of v^1_1 in W^j. Now, $\phi(\mathcal{W}) = \mathcal{W} \leftrightarrow^{i,j}_{a,b}$.

C. *if the vertex v^1_1 occurs only in the walk W^1:*
   By the case (A), the vertex v^1_1 occurs multiple times in W^1. Let b be the index of the last occurrence of v^1_1 in W^1 and a be the index of the second last occurrence of v^1_1 in W^1. Note that $a = 1$ if v^1_1 occurs only twice in W^1, and note also that $1 \leq a \leq b-2$.

   1. *if $r^1_1 = r^1_b = 0$:*
      (a) *if $W^1[2, b-1]$ is not a palindrome, then $\phi(\mathcal{W}) = (W^1\overleftarrow{[2, b-1]}, W^2, \dots, W^p)$.*
      (b) *otherwise, if $b < \ell_1$, then $\phi(\mathcal{W}) = W^1[1,b] \diamond \phi(W^1[b+1, \ell_1], W^2, \dots, W^p)$.*
      (c) *otherwise (i.e., $b = \ell_1$), $\phi(\mathcal{W}) = W^1 \sqcup \phi(W^2, \dots, W^p)$.*
   2. *if the index b is not a digon in W^1, then $\phi(\mathcal{W}) = \mathcal{W} \frown^{1,1}_{1,b}$.*
      Note: If neither case (1) nor (2) applies, then $r^1_1 \neq 0$.
   3. *if $W^1[2, a-1]$ is not a palindrome, then $\phi(\mathcal{W}) = (W^1\overleftarrow{[2, a-1]}, W^2, \dots, W^p)$.*
      Note: If $a = 1$, then $W^1[2, a-1]$ is the empty walk which is a palindrome.
   4. *if $v^1_{a+1} = v^1_{b-1}$:*
      (a) *if $W^1[a+1, b-1]$ is not a palindrome, then $\phi(\mathcal{W}) = (W^1\overleftarrow{[a+1, b-1]}, W^2, \dots, W^p)$.*
      (b) *otherwise, $\phi(\mathcal{W}) = W^1[1,b] \diamond \phi(W^1[b+1, \ell_1], W^2, \dots, W^p)$.*
      Note: Here $W^1[b+1, \ell_1]$ cannot be an empty walk because by case (C.2) b is a digon in W^1.

   X. *otherwise, $\phi(\mathcal{W}) = W^1[1,a] \diamond \phi(W^1[a+1, \ell_1], W^2, \dots, W^p)$.*
      Note: The case C.X will form a "common case" with the case D.X.

D. *if the vertex v^1_1 occurs in exactly two different walks:*
   Let i be the index of the another walk W^i in which v^1_1 occurs and let b be the index of the first occurrence of v^1_1 in W^i.

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   3. *if v^1_1 occurs at least twice in W^i, then let c be the index of its second occurrence and $\phi(\mathcal{W}) = \mathcal{W} \leftrightarrow^{1,i}_{2,c+1}$.*
      Note: It can happen that one of the suffixes in this case is empty. However, both of them cannot be empty at the same time because W^1 and W^i have different ending vertices because $\mathcal{W}$ is proper.
      Note: In the remaining cases, v^1_1 occurs exactly once in W^i, and this occurrence is a digon at index b.
      Now, let a be the index of the last occurrence of v^1_1 in W^1 (if v^1_1 occurs only once in W^1, then $a = 1$).
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Cancellation arguments

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- Good news: We managed to construct ϕ, but it is very complicated
  - Single-path version has 8 cases, multi-path version 18 cases

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$\Rightarrow$ By DeMillo–Lipton–Schwartz–Zippel lemma, $2^k n^{\mathcal{O}(1)}$ time algorithm for k-colored (s, t)-path that works with high probability

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- The proof that if no solution exists then $f(\mathcal{C}_\ell)$ is identically zero is the most complicated part
- Open problem: Is there an FPT-algorithm for maximum colored two disjoint paths?
- Open problem: Is there a $1.99^k n^{\mathcal{O}(1)}$ time algorithm for longest (s, t)-path?

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