Minor Containment and Disjoint Paths in almost-linear time

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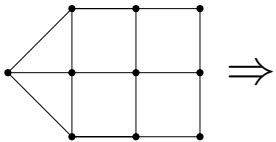

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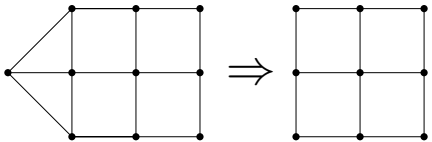

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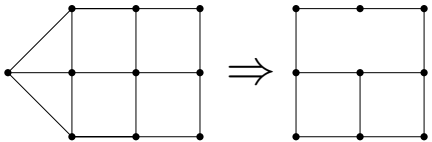

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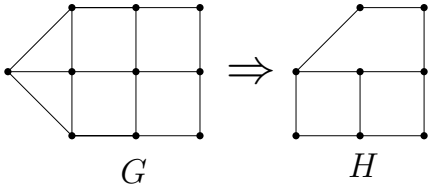

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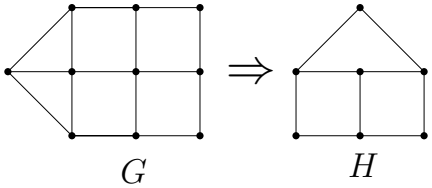

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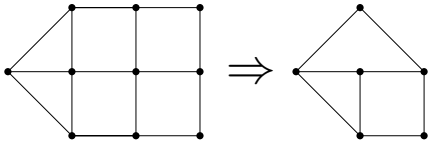

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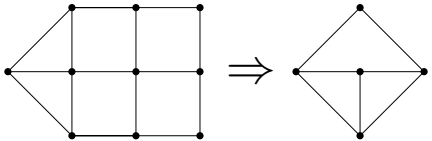

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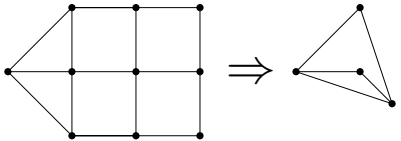

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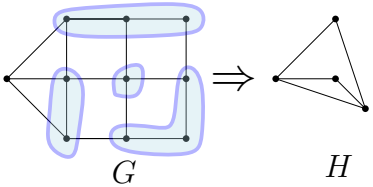

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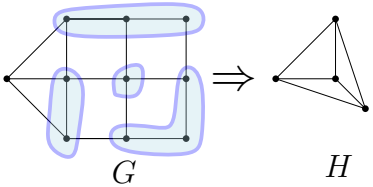

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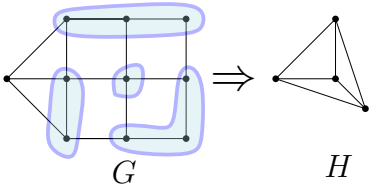

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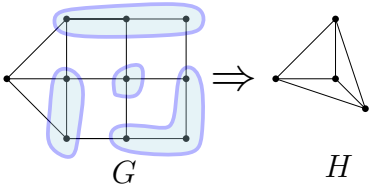

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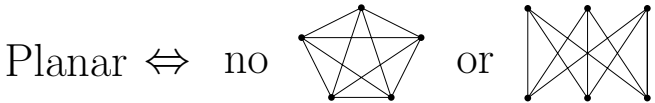

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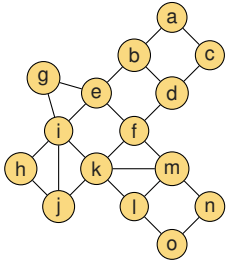

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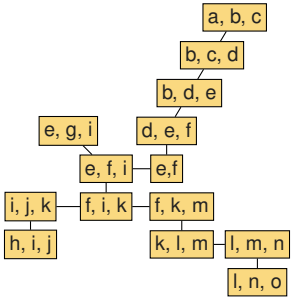

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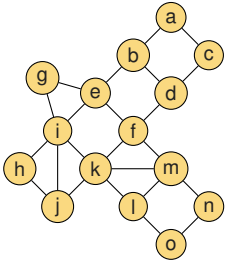

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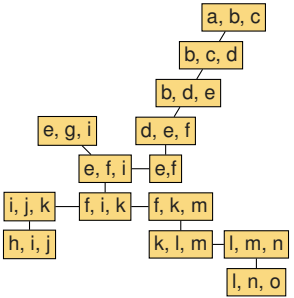

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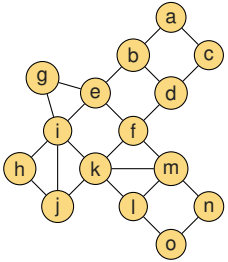

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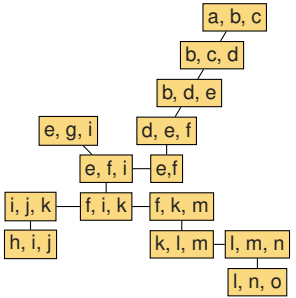

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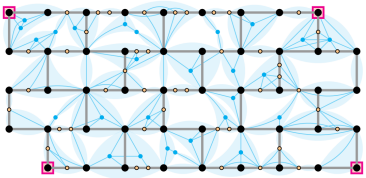

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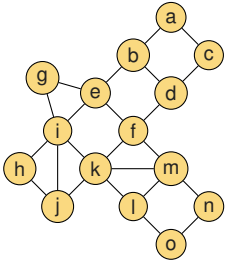
- Robertson & Seymour: Detect irrelevant vertex in $f(H) \cdot n^2$ time $\Rightarrow f(H) \cdot n^3$ time algorithm



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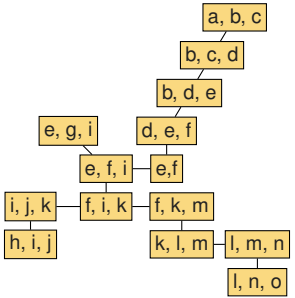

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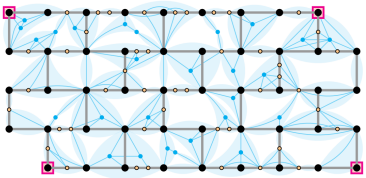

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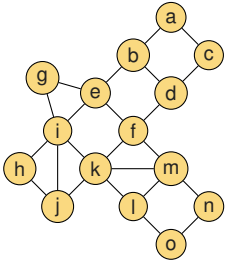
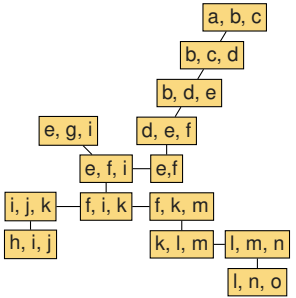
- If treewidth of G is $\leq f(H)$, solve the problem by dynamic programming in $f(H) \cdot n$ time
- If treewidth is $> f(H)$, detect and remove an Irrelevant Vertex from G

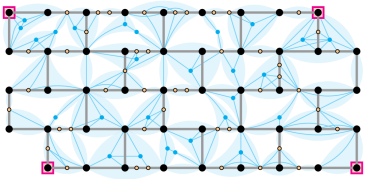


Graph G
Treewidth 2



A tree decomposition of G
Width = 2

- Robertson & Seymour: Detect irrelevant vertex in $f(H) \cdot n^2$ time $\Rightarrow$ $f(H) \cdot n^3$ time algorithm
- Kawarabayashi, Kobayashi & Reed: Detect irrelevant vertex in $f(H) \cdot n$ time $\Rightarrow$ $f(H) \cdot n^2$ time algorithm



Outline of our algorithm

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1. Fast implementation of the irrelevant vertex technique on apex-minor-free graphs

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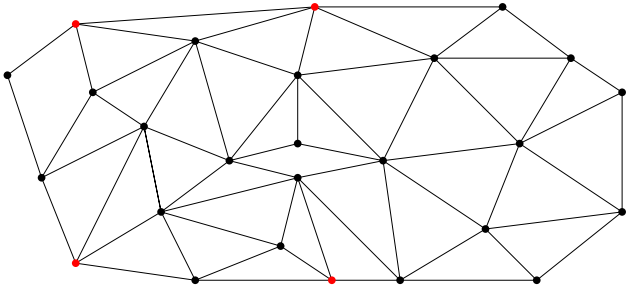
4. Reducing general graphs to clique-minor-free graphs

Algorithm for apex-minor-free graphs



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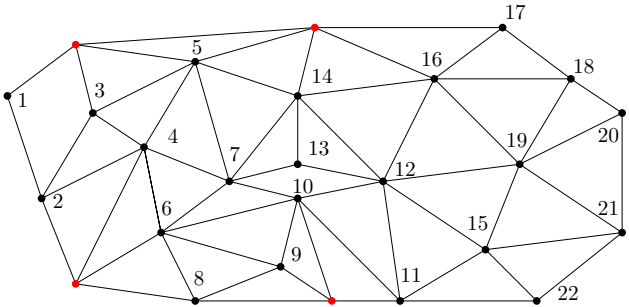

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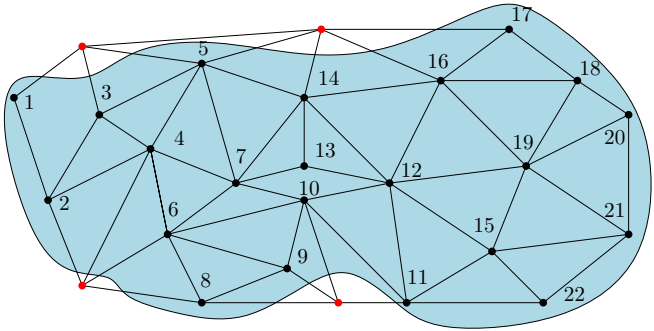

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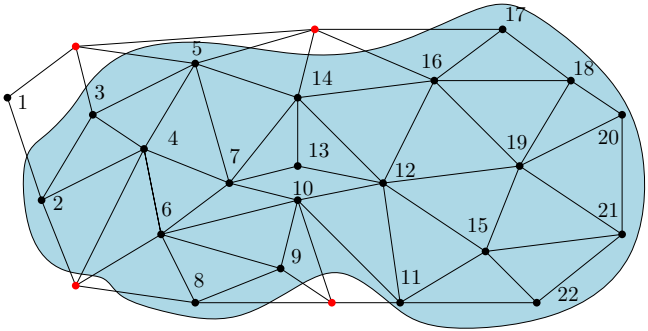

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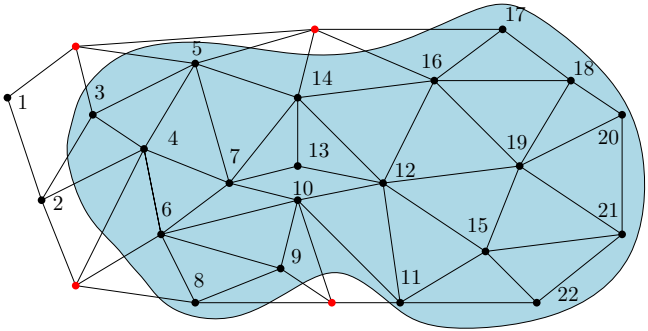

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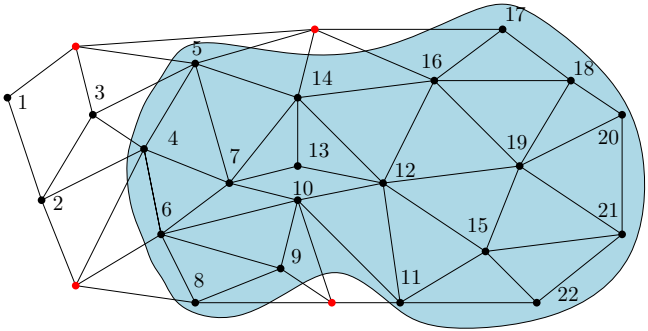

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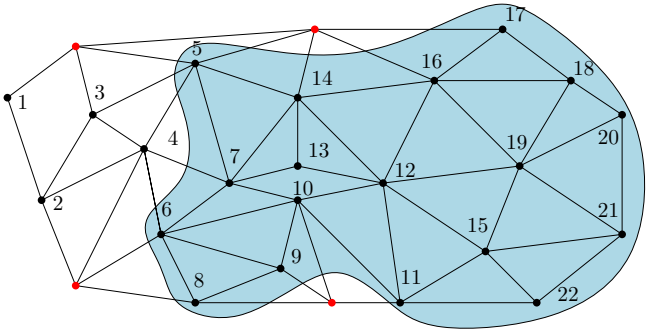

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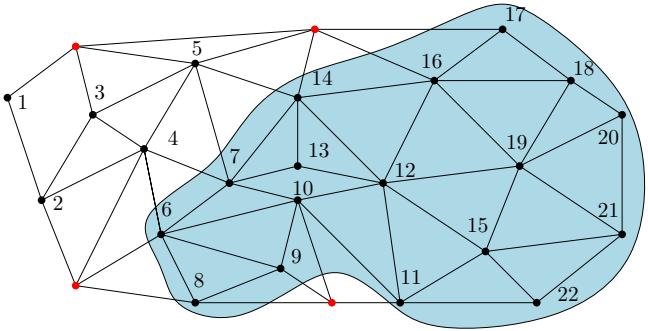

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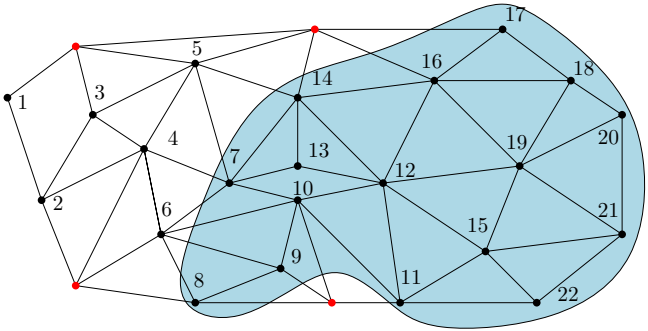

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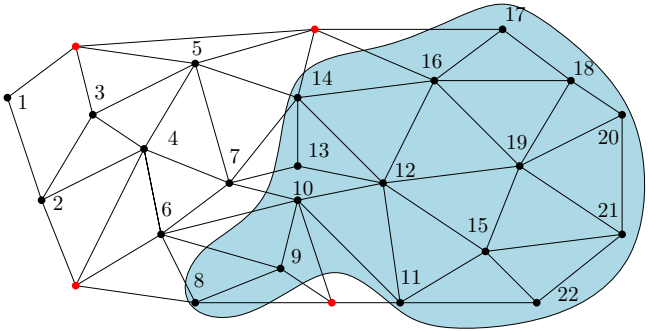

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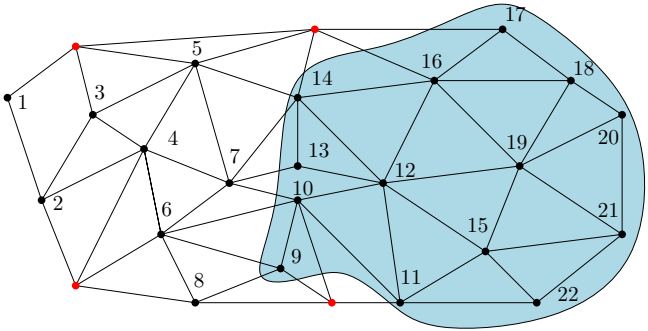

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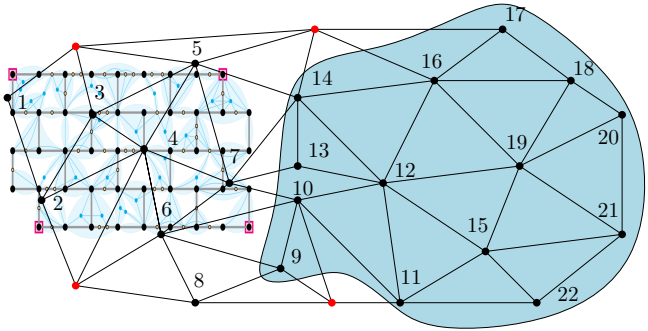

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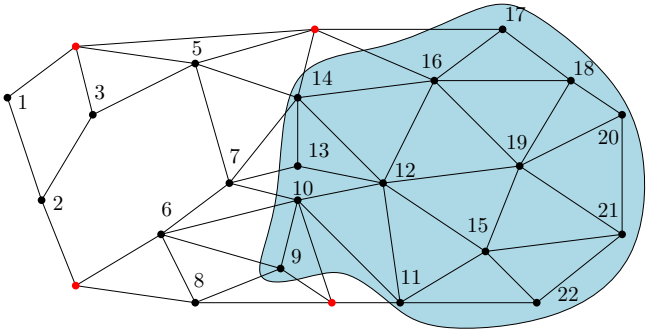

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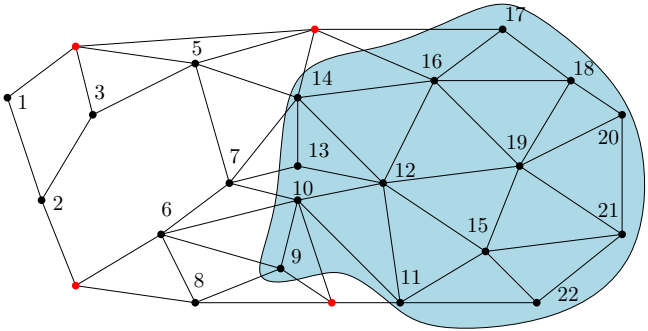

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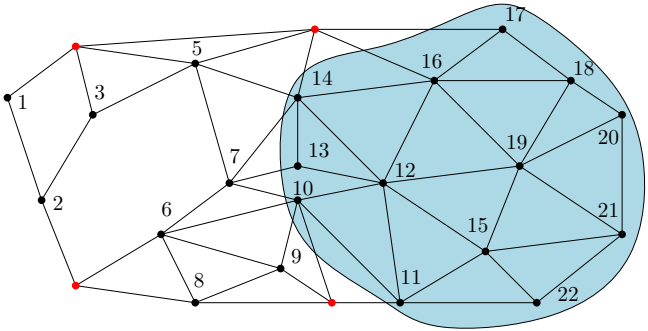

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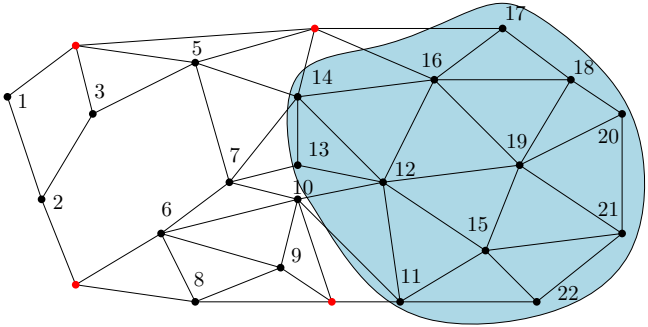

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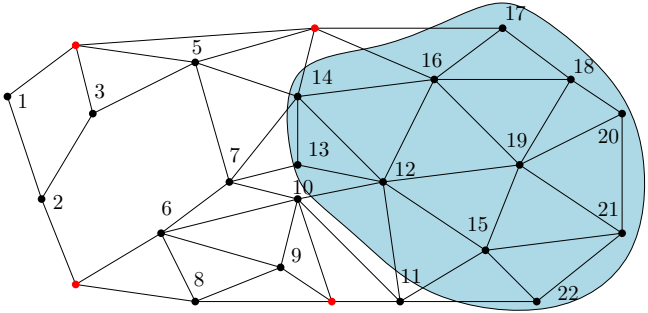

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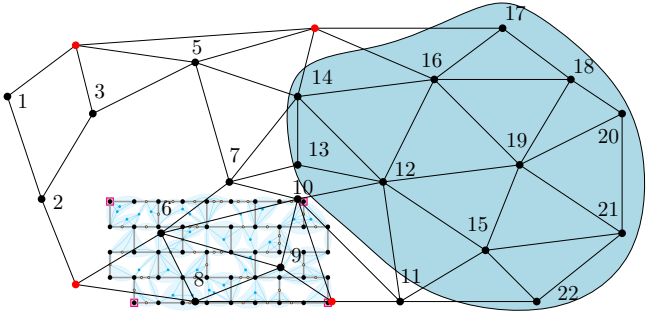

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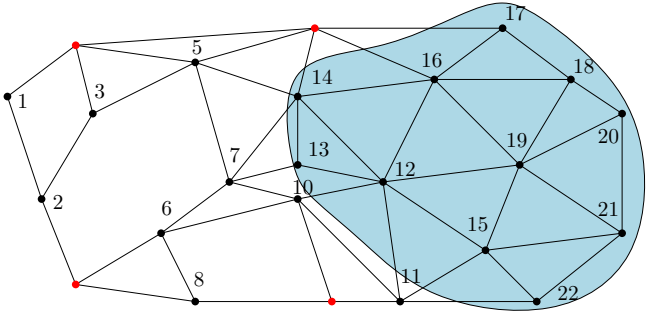

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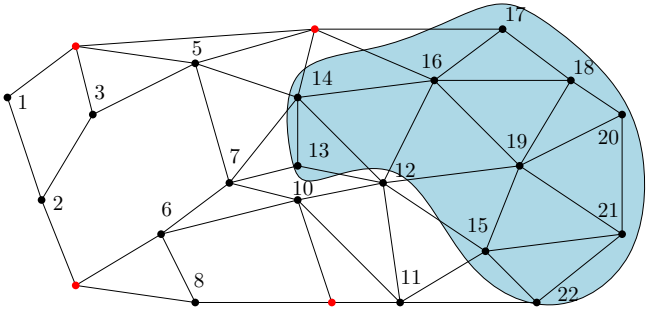

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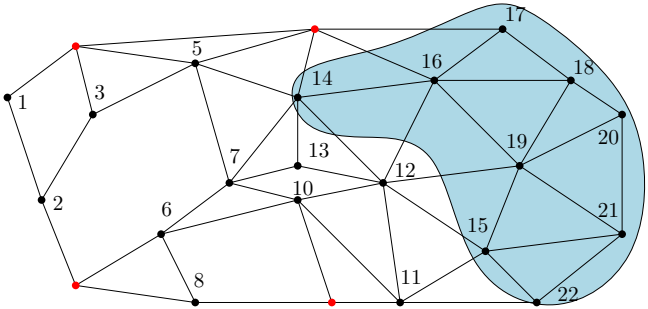

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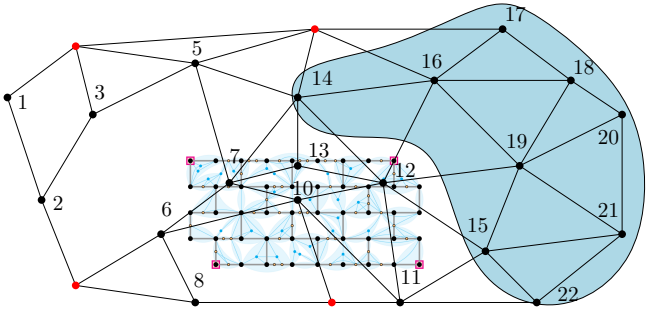

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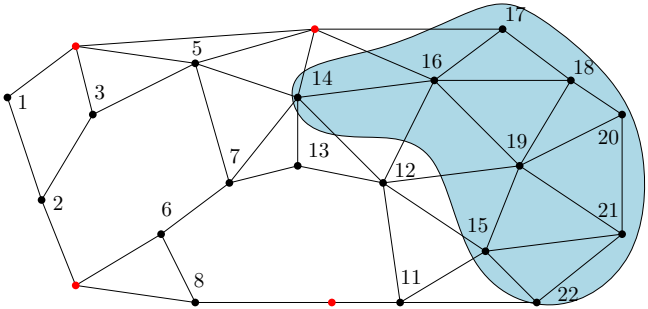

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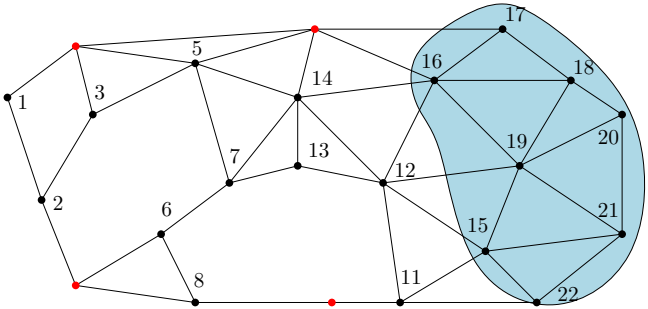

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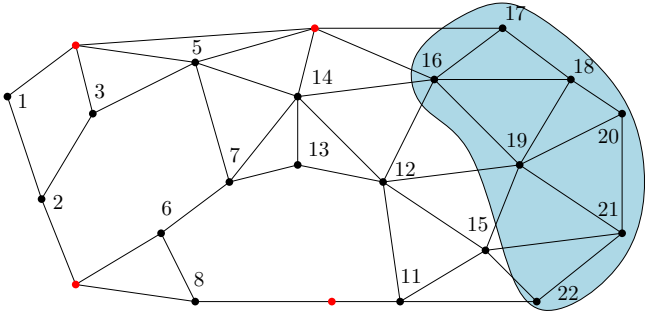

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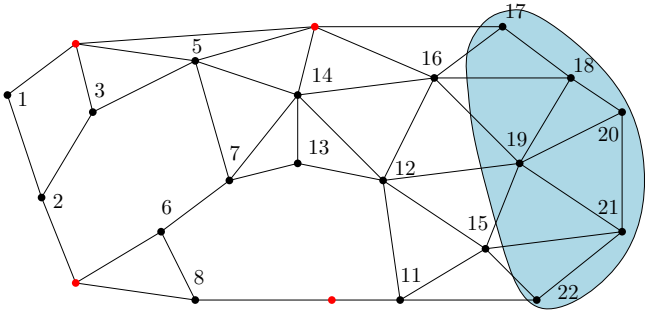

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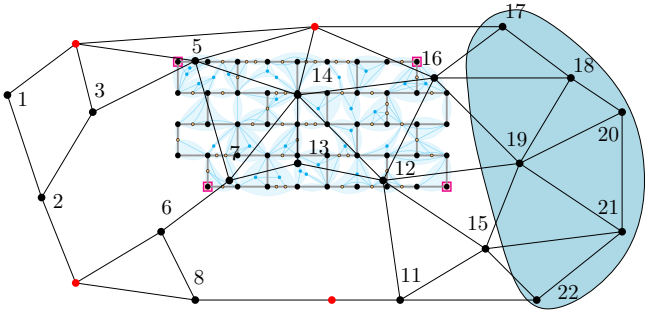

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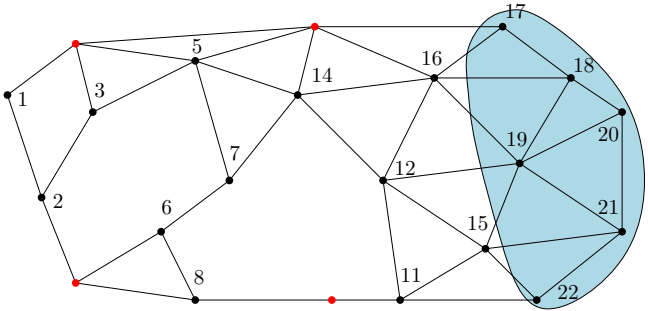

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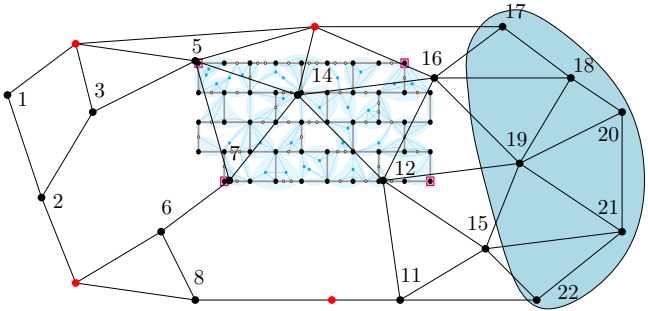

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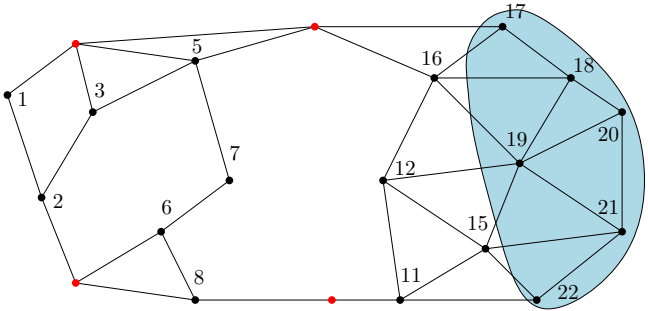

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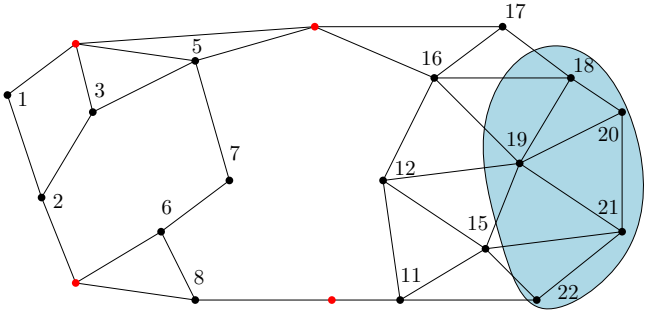

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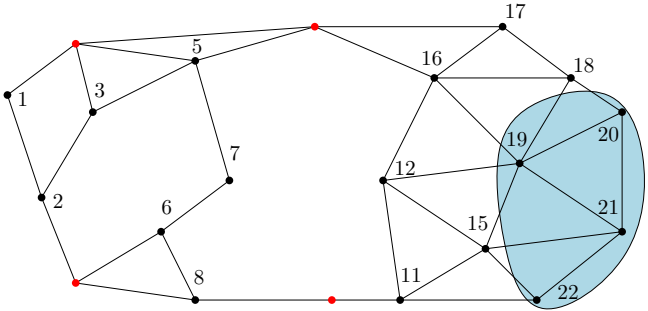

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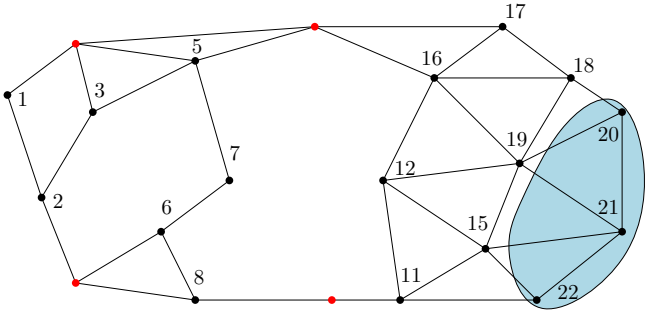

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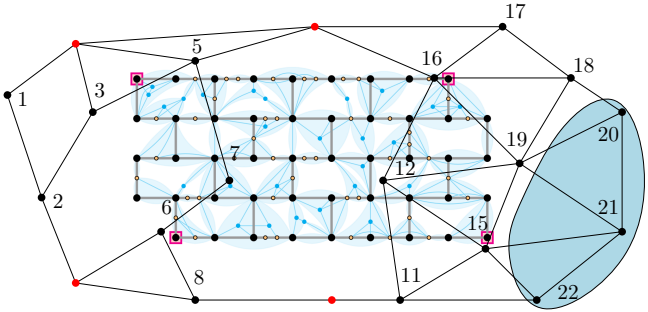

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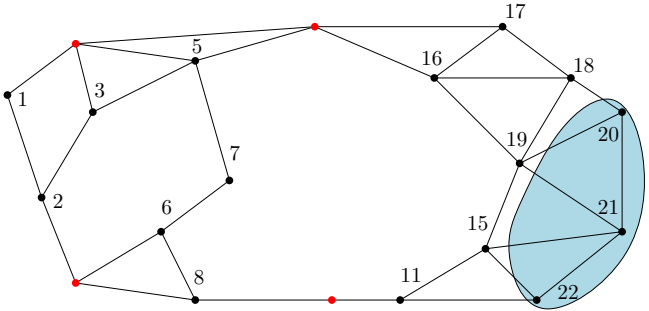

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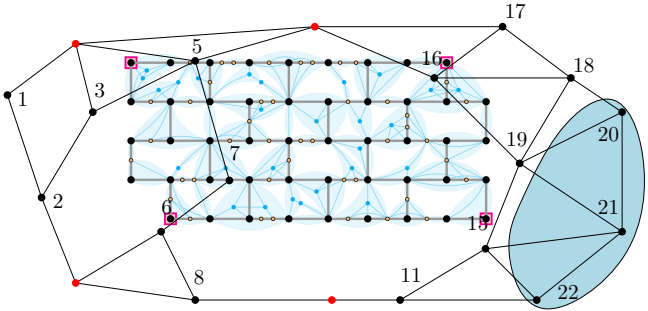

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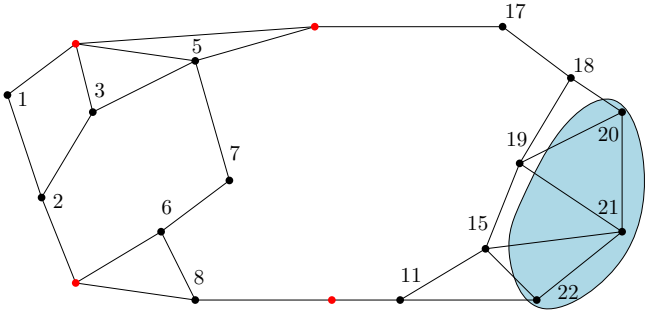

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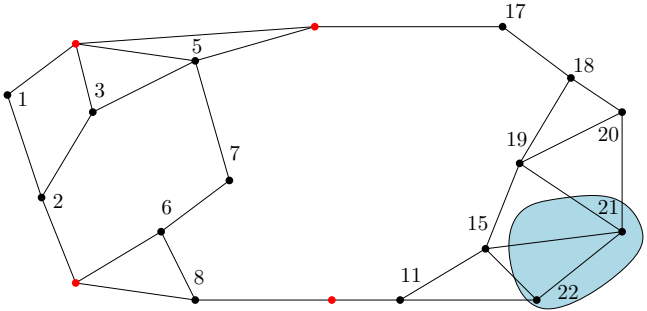

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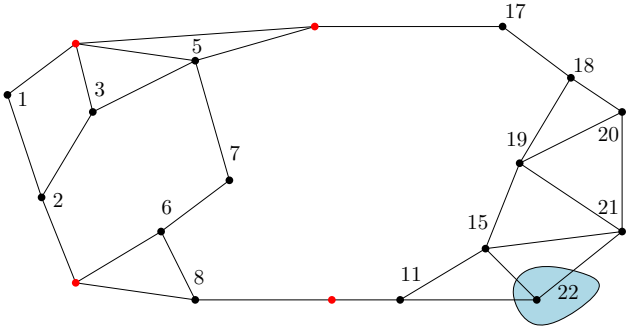

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Thank you!