Grid Induced Minor Theorem for Graphs of Small Degree

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UNIVERSITY OF BERGEN

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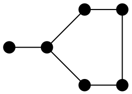
Graph containment



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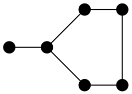
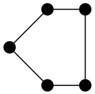


1. Induced subgraph
   - vertex deletions



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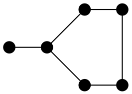

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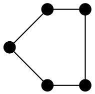

2. Induced minor
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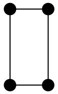

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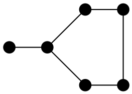

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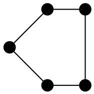

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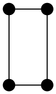

3. Minor
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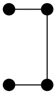

Graph classses defined by containment

- For a graph H, we can define graph classes by excluding H
  - H-minor-free graphs
  - H-induced-minor-free graphs

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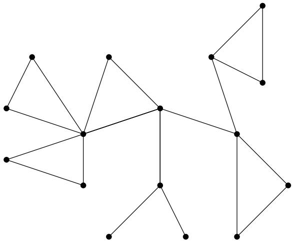
- Example: C_4-minor-free graphs

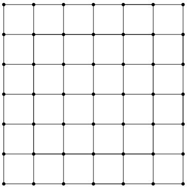


- Every biconnected component is a triangle
- Chordal and treewidth ≤ 2

Grid minor theorem

Robertson & Seymour, Graph Minors V. ('86):

- If H is planar, then H-minor-free graphs have bounded treewidth



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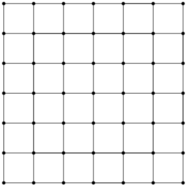
$\Leftrightarrow$ There is a function f so that if a graph has treewidth $\geq f(k)$, then it contains a $k \times k$-grid as a minor



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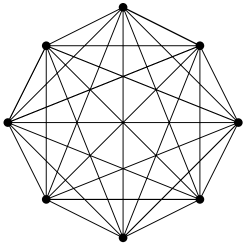
- Complete graph contains only complete graphs as induced minors



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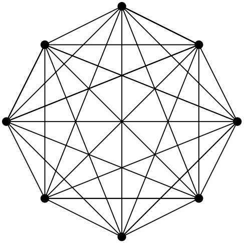
⇒ C_4-induced-minor-free graphs have unbounded treewidth



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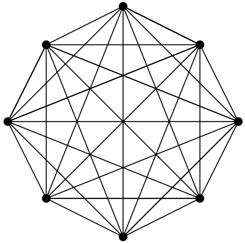
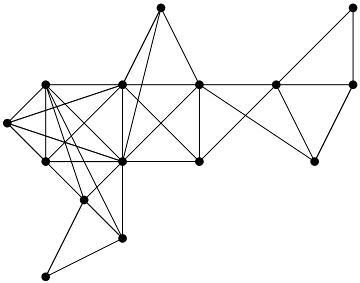
⇒ C_4-induced-minor-free graphs have unbounded treewidth

- C_4-induced-minor-free graphs ⇔ chordal graphs





When does grid minor theorem work for induced minors?

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Theorem (Fomin, Golovach, and Thilikos, 2011)

For any graph H there is a constant C_H so that any H-minor-free graph with treewidth $\geq C_H \cdot k$ contains a $k \times k$-grid as an induced minor.

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Theorem (K. 2022)

There is a function $f(k, d) = \mathcal{O}(k^{10} + 2^{d^5})$ so that any graph with maximum degree d and treewidth $\geq f(k, d)$ contains a $k \times k$-grid as an induced minor.

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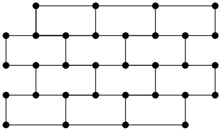
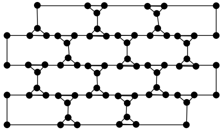
Corollary (Induced subgraph version)

There is a function $f(k, d)$ so that any graph with maximum degree d and treewidth $\geq f(k, d)$ contains a $k \times k$-wall or the line graph of a $k \times k$-wall as an induced subgraph.





Conjectured by Aboulker, Adler, Kim, Sintiari, and Trotignon, 2021

Algorithmic motivation/application

Open problem

Is there a quasipolynomial time algorithm for maximum independent set on H-induced-minor-free graphs when H is planar?

Recall: Quasipolynomial time is $2^{\text{polylog} n}$ time

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Solved special cases:

- $H = P_t$ [Gartland and Lokshtanov, 2020]
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- $H = W_4$, K_5^-, or $K_{2,q}$ [Dallard, Milanic, Storgel, 2022]

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- even-hole-free graphs

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$\Rightarrow$ Linear-time algorithm when input graphs have bounded degree

$\Rightarrow$ $\mathcal{O}(2^{n/\log^{1/6} n})$ time algorithm in general

Proof

Sparsifiable vertices

A vertex v of G is *sparsifiable* if it satisfies one of the following:

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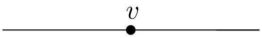

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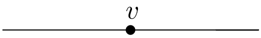
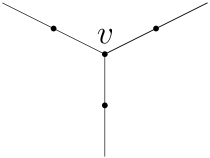


2. v has degree 3 and all neighbors of v have degree ≤ 2



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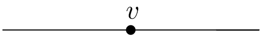

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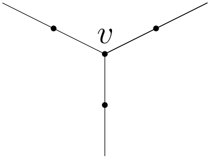
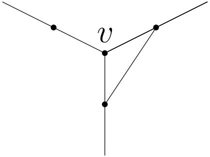


3. *v* has degree 3, one of its neighbors has degree ≤ 2, and two others form a triangle with *v*



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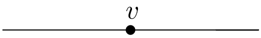

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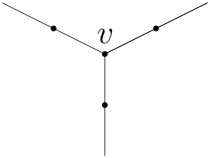

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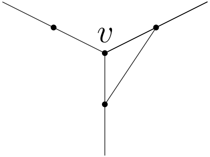

Lemma

Let G, H be graphs so that every vertex of G is sparsifiable and H has minimum degree ≥ 3. Then G contains H as an induced minor if G contains H as a minor.

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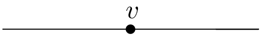

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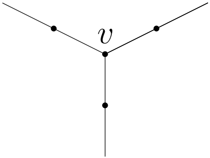

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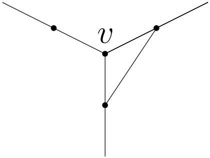

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Lemma

Let G, H be graphs so that every vertex of G is sparsifiable and H has minimum degree ≥ 3. Then G contains H as an induced minor if G contains H as a minor.

Goal: Make every vertex of G sparsifiable while maintaining high treewidth

Iterative sparsification

Let G be a graph and d the maximum degree of G

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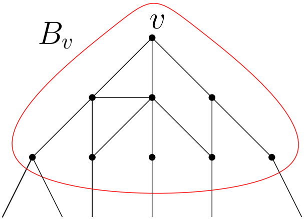
$\Rightarrow$ Finish by using the lemma of the previous slide with the grid minor theorem

Making vertices sparsifiable

For each vertex $v \in I_i$, let B_v be the distance-2 ball around v



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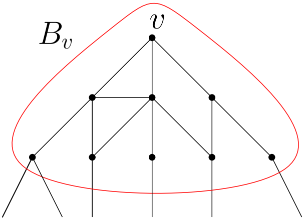

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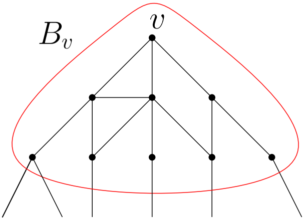

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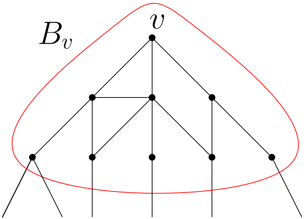

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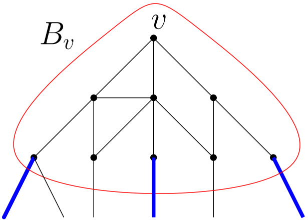

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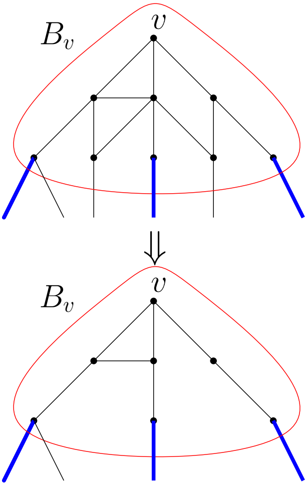

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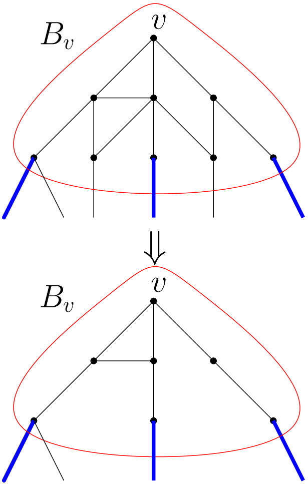

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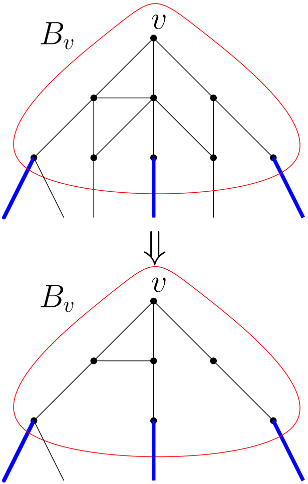

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- G'' is a minor of G, with a minor model given by the balls B_v
- delete vertices from B_v so that v is either deleted or made sparsifiable while preserving the minor model(case analysis with 4 cases)
- all vertices in I_i become sparsifiable while preserving G'' as a minor

Thank you!

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