An Improved Parameterized Algorithm for Treewidth

Tuukka Korhonen

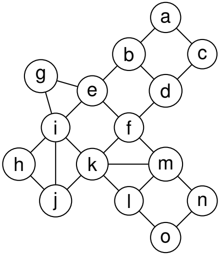
UNIVERSITY OF BERGEN

joint work with
Daniel Lokshtanov, UCSB

Aalto TCS Seminar

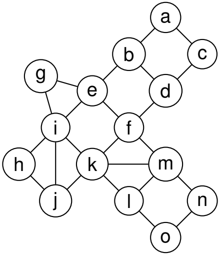
14 December 2022

Treewidth

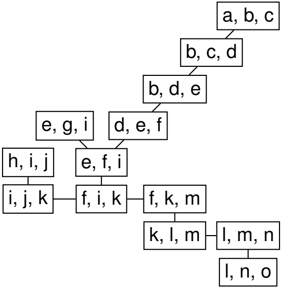


Graph G

Treewidth

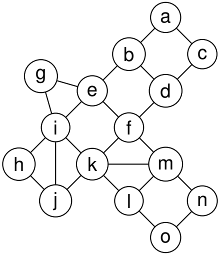


Graph G

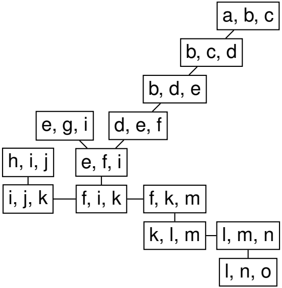


A tree decomposition of G

Treewidth



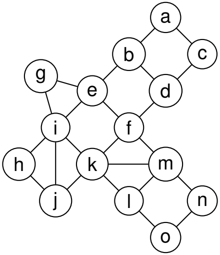
Graph G



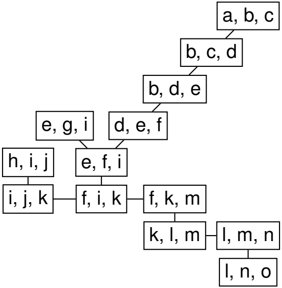
A tree decomposition of G

1. Every vertex should be in a bag

Treewidth



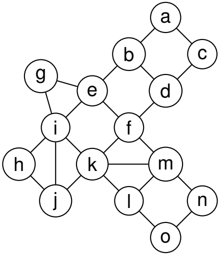
Graph G



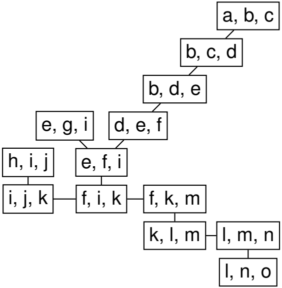
A tree decomposition of G

1. Every vertex should be in a bag
2. Every edge should be in a bag

Treewidth



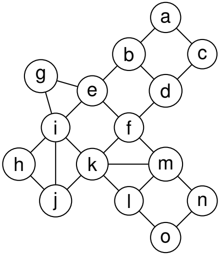
Graph G



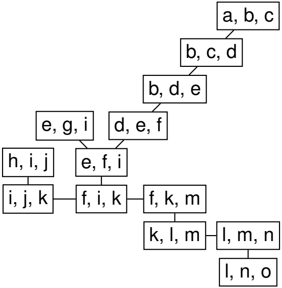
A tree decomposition of G

1. Every vertex should be in a bag
2. Every edge should be in a bag
3. Bags containing a vertex should form a connected subtree

Treewidth



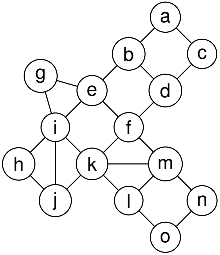
Graph G



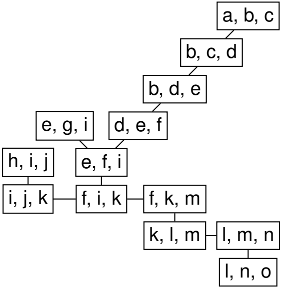
A tree decomposition of G

1. Every vertex should be in a bag
2. Every edge should be in a bag
3. Bags containing a vertex should form a connected subtree
4. Width = maximum bag size -1

Treewidth



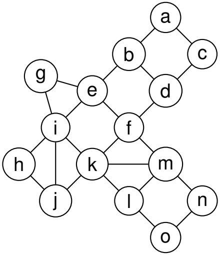
Graph G



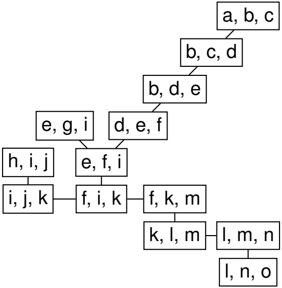
A tree decomposition of G
Width = 2

1. Every vertex should be in a bag
2. Every edge should be in a bag
3. Bags containing a vertex should form a connected subtree
4. Width = maximum bag size -1

Treewidth



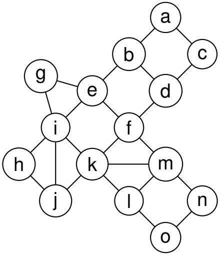
Graph G



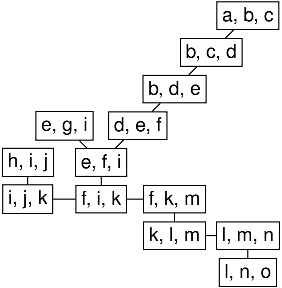
A tree decomposition of G
Width = 2

1. Every vertex should be in a bag
2. Every edge should be in a bag
3. Bags containing a vertex should form a connected subtree
4. Width = maximum bag size -1
5. Treewidth of G = minimum width of tree decomposition of G

Treewidth



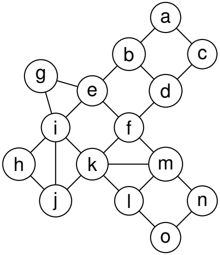
Graph G
Treewidth 2



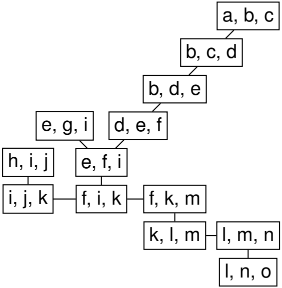
A tree decomposition of G
Width = 2

1. Every vertex should be in a bag
2. Every edge should be in a bag
3. Bags containing a vertex should form a connected subtree
4. Width = maximum bag size -1
5. Treewidth of G = minimum width of tree decomposition of G

Treewidth



Graph G
Treewidth 2



A tree decomposition of G
Width = 2

1. Every vertex should be in a bag
2. Every edge should be in a bag
3. Bags containing a vertex should form a connected subtree
4. Width = maximum bag size -1
5. Treewidth of G = minimum width of tree decomposition of G

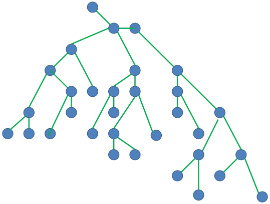
[Bertele & Brioschi'72, Halin'76, Robertson & Seymour'84]

Treewidth of graphs

Some graphs of small treewidth:

Treewidth of graphs

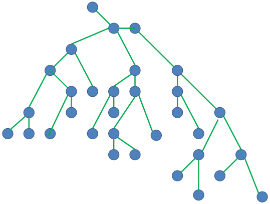
Some graphs of small treewidth:



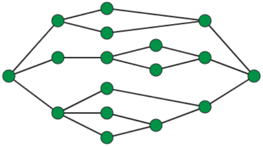
Trees (tw $\leq$ 1)

Treewidth of graphs

Some graphs of small treewidth:



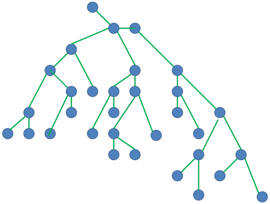
Trees (tw $\leq$ 1)



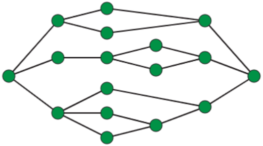
Series-parallel (tw $\leq$ 2)

Treewidth of graphs

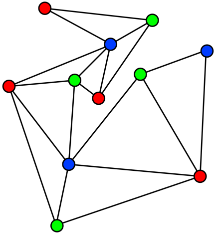
Some graphs of small treewidth:



Trees (tw ≤ 1)



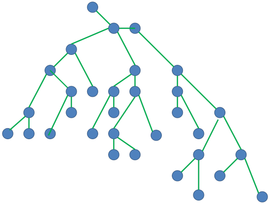
Series-parallel (tw ≤ 2)



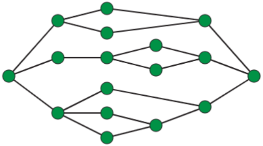
k-outerplanar (tw $\leq 3k-1$)

Treewidth of graphs

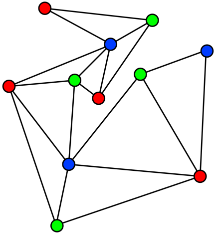
Some graphs of small treewidth:



Trees (tw ≤ 1)



Series-parallel (tw ≤ 2)

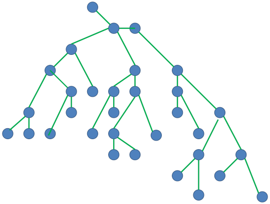


k-outerplanar (tw $\leq 3k - 1$)

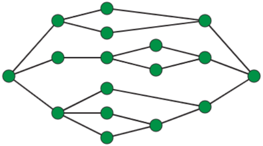
Some graphs of large treewidth:

Treewidth of graphs

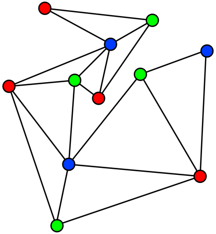
Some graphs of small treewidth:



Trees (tw ≤ 1)

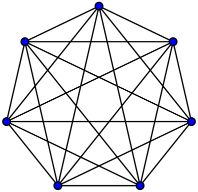


Series-parallel (tw ≤ 2)



k-outerplanar (tw $\leq 3k - 1$)

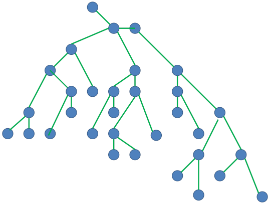
Some graphs of large treewidth:



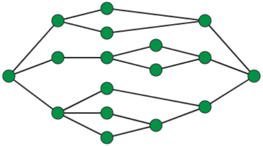
Clique (tw $= n - 1$)

Treewidth of graphs

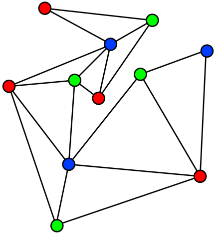
Some graphs of small treewidth:



Trees (tw ≤ 1)

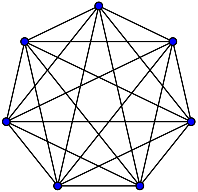


Series-parallel (tw ≤ 2)

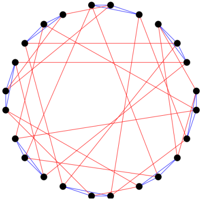


k-outerplanar (tw $\leq 3k - 1$)

Some graphs of large treewidth:



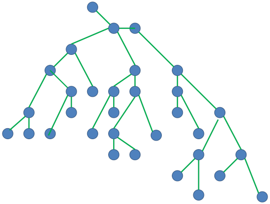
Clique (tw $= n - 1$)



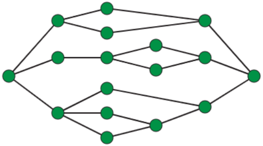
Expanders (tw $= \Theta(n)$)

Treewidth of graphs

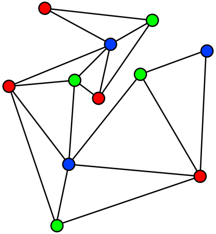
Some graphs of small treewidth:



Trees (tw ≤ 1)

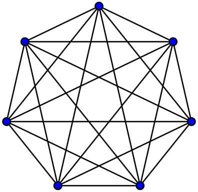


Series-parallel (tw ≤ 2)

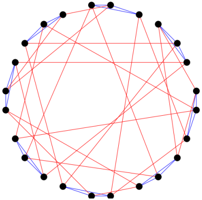


k-outerplanar (tw $\leq 3k - 1$)

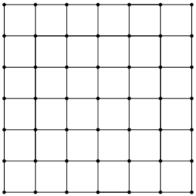
Some graphs of large treewidth:



Clique (tw $= n - 1$)

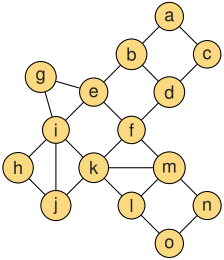


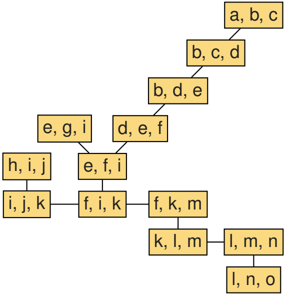
Expanders (tw $= \Theta(n)$)



$n \times n$-grid (tw $= n$)

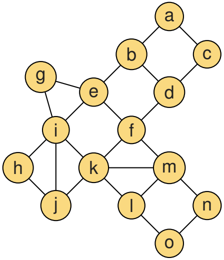
Why treewidth: Algorithms

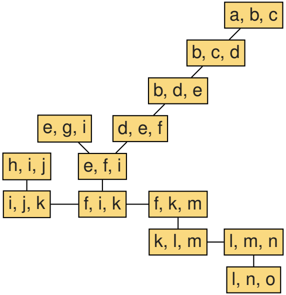




Why treewidth: Algorithms

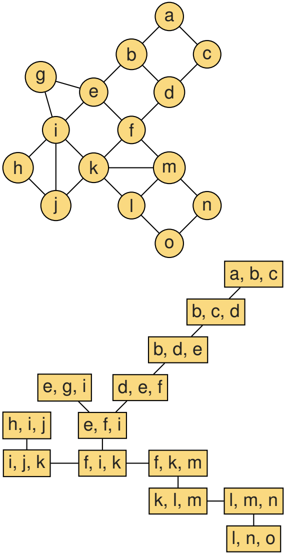
- Algorithms on trees generalize to algorithms on graphs of small treewidth





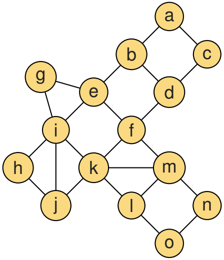
Why treewidth: Algorithms

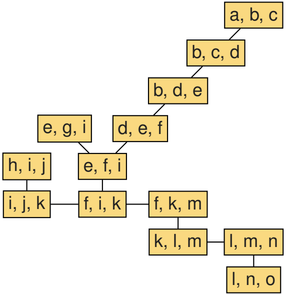
- Algorithms on trees generalize to algorithms on graphs of small treewidth
- Given a graph with a tree decomposition of width k:



Why treewidth: Algorithms

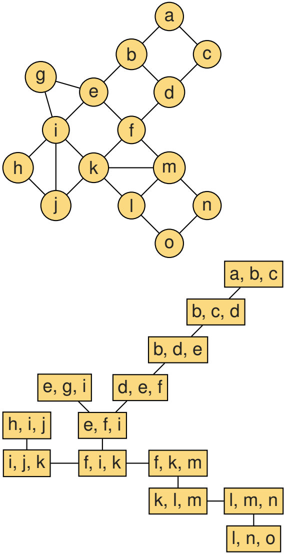
- Algorithms on trees generalize to algorithms on graphs of small treewidth
- Given a graph with a tree decomposition of width k:
- Maximum independent set in time $\mathcal{O}(2^k \cdot n)$





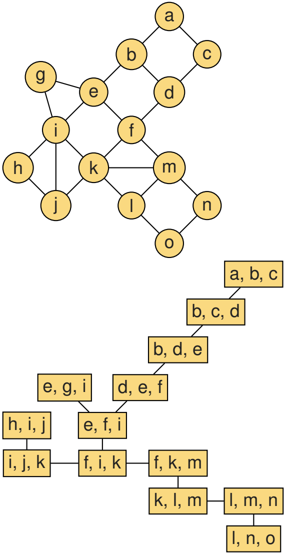
Why treewidth: Algorithms

- Algorithms on trees generalize to algorithms on graphs of small treewidth
- Given a graph with a tree decomposition of width k:
- Maximum independent set in time $\mathcal{O}(2^k \cdot n)$
- Minimum dominating set in time $\mathcal{O}(3^k \cdot n)$



Why treewidth: Algorithms

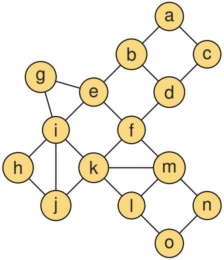
- Algorithms on trees generalize to algorithms on graphs of small treewidth
- Given a graph with a tree decomposition of width k:
- Maximum independent set in time $\mathcal{O}(2^k \cdot n)$
- Minimum dominating set in time $\mathcal{O}(3^k \cdot n)$
- Hamiltonian cycle in time $2^{\mathcal{O}(k)} \cdot n$

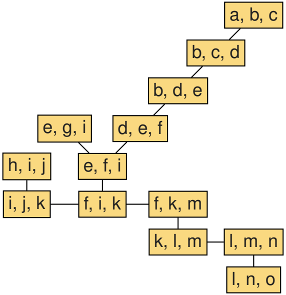


Why treewidth: Algorithms

- Algorithms on trees generalize to algorithms on graphs of small treewidth

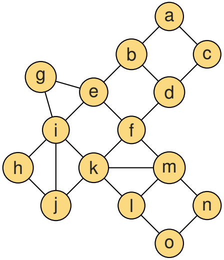
- Given a graph with a tree decomposition of width k:
- Maximum independent set in time $\mathcal{O}(2^k \cdot n)$
- Minimum dominating set in time $\mathcal{O}(3^k \cdot n)$
- Hamiltonian cycle in time $2^{\mathcal{O}(k)} \cdot n$
- Any problem in MSO-logic in time $f(k) \cdot n$

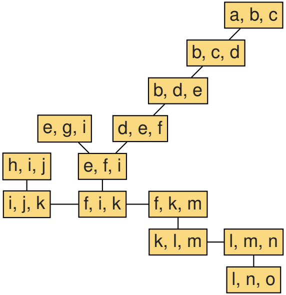




Why treewidth: Algorithms

- Algorithms on trees generalize to algorithms on graphs of small treewidth
- Given a graph with a tree decomposition of width k:
- Maximum independent set in time $\mathcal{O}(2^k \cdot n)$
- Minimum dominating set in time $\mathcal{O}(3^k \cdot n)$
- Hamiltonian cycle in time $2^{\mathcal{O}(k)} \cdot n$
- Any problem in MSO-logic in time $f(k) \cdot n$
- Need to compute the tree decomposition first!





Computing treewidth (and the tree decomposition)

Computing treewidth (and the tree decomposition)

- NP-complete [Arnborg, Corneil, Proskurowski '87]

Computing treewidth (and the tree decomposition)

- NP-complete [Arnborg, Corneil, Proskurowski '87]
- $\mathcal{O}(n^{k+2})$ time [Arnborg, Corneil, Proskurowski '87]

Computing treewidth (and the tree decomposition)

- NP-complete [Arnborg, Corneil, Proskurowski ’87]
- $\mathcal{O}(n^{k+2})$ time [Arnborg, Corneil, Proskurowski ’87]
- $f(k) \cdot n^2$ time, non-constructive [Robertson & Seymour’86]

Computing treewidth (and the tree decomposition)

- NP-complete [Arnborg, Corneil, Proskurowski '87]
- $\mathcal{O}(n^{k+2})$ time [Arnborg, Corneil, Proskurowski '87]
- $f(k) \cdot n^2$ time, non-constructive [Robertson & Seymour'86]
- $2^{\mathcal{O}(k^3)} n \log^2 n$ time [Bodlaender & Kloks, Lagergren & Arnborg, '91]

Computing treewidth (and the tree decomposition)

- NP-complete [Arnborg, Corneil, Proskurowski '87]
- $\mathcal{O}(n^{k+2})$ time [Arnborg, Corneil, Proskurowski '87]
- $f(k) \cdot n^2$ time, non-constructive [Robertson & Seymour'86]
- $2^{\mathcal{O}(k^3)} n \log^2 n$ time [Bodlaender & Kloks, Lagergren & Arnborg, '91]
  - Using $k^{\mathcal{O}(k)} n \log^2 n$ time 8-approximation of [Lagergren '90]

Computing treewidth (and the tree decomposition)

- NP-complete [Arnborg, Corneil, Proskurowski '87]
- $\mathcal{O}(n^{k+2})$ time [Arnborg, Corneil, Proskurowski '87]
- $f(k) \cdot n^2$ time, non-constructive [Robertson & Seymour'86]
- $2^{\mathcal{O}(k^3)} n \log^2 n$ time [Bodlaender & Kloks, Lagergren & Arnborg, '91]
  - Using $k^{\mathcal{O}(k)} n \log^2 n$ time 8-approximation of [Lagergren '90]
- $2^{\mathcal{O}(k^3)} n$ time [Bodlaender '93]

Computing treewidth (and the tree decomposition)

- NP-complete [Arnborg, Corneil, Proskurowski '87]
- $\mathcal{O}(n^{k+2})$ time [Arnborg, Corneil, Proskurowski '87]
- $f(k) \cdot n^2$ time, non-constructive [Robertson & Seymour'86]
- $2^{\mathcal{O}(k^3)} n \log^2 n$ time [Bodlaender & Kloks, Lagergren & Arnborg, '91]
  - Using $k^{\mathcal{O}(k)} n \log^2 n$ time 8-approximation of [Lagergren '90]
- $2^{\mathcal{O}(k^3)} n$ time [Bodlaender '93]
- "Can the dependence $2^{\mathcal{O}(k^3)}$ on k be improved?" [Downey & Fellows'99], [Telle'06], [Bodlaender, Drange, Dregi, Fomin, Lokshtanov & Pilipczuk'16], [Bodlaender, Jaffke & Telle'20]

Computing treewidth (and the tree decomposition)

- NP-complete [Arnborg, Corneil, Proskurowski '87]
- $\mathcal{O}(n^{k+2})$ time [Arnborg, Corneil, Proskurowski '87]
- $f(k) \cdot n^2$ time, non-constructive [Robertson & Seymour'86]
- $2^{\mathcal{O}(k^3)} n \log^2 n$ time [Bodlaender & Kloks, Lagergren & Arnborg, '91]
  - Using $k^{\mathcal{O}(k)} n \log^2 n$ time 8-approximation of [Lagergren '90]
- $2^{\mathcal{O}(k^3)} n$ time [Bodlaender '93]
- "Can the dependence $2^{\mathcal{O}(k^3)}$ on k be improved?" [Downey & Fellows'99], [Telle'06], [Bodlaender, Drange, Dregi, Fomin, Lokshtanov & Pilipczuk'16], [Bodlaender, Jaffke & Telle'20]

Theorem (This work)

There is a $2^{\mathcal{O}(k^2)} n^4$ time algorithm for treewidth.

Approximating treewidth

- Polynomial-time approximation:

Approximating treewidth

- Polynomial-time approximation:
  - $\mathcal{O}(\sqrt{\log(k)})$-approximation [Feige, Hajiaghayi & Lee'08]

Approximating treewidth

- Polynomial-time approximation:
  - $\mathcal{O}(\sqrt{\log(k)})$-approximation [Feige, Hajiaghayi & Lee'08]
  - Constant-factor approximation NP-hard for any constant, assuming SSE-conjecture [Wu, Austrin, Pitassi & Liu'14]

Approximating treewidth

- Polynomial-time approximation:
  - $\mathcal{O}(\sqrt{\log(k)})$-approximation [Feige, Hajiaghayi & Lee'08]
  - Constant-factor approximation NP-hard for any constant, assuming SSE-conjecture [Wu, Austrin, Pitassi & Liu'14]
- FPT-approximation:

Approximating treewidth

- Polynomial-time approximation:
  - $\mathcal{O}(\sqrt{\log(k)})$-approximation [Feige, Hajiaghayi & Lee'08]
  - Constant-factor approximation NP-hard for any constant, assuming SSE-conjecture [Wu, Austrin, Pitassi & Liu'14]
- FPT-approximation:
  - $2^{\mathcal{O}(k)}n^2$ time 4-approximation [Robertson & Seymour'86]

Approximating treewidth

- Polynomial-time approximation:
  - $\mathcal{O}(\sqrt{\log(k)})$-approximation [Feige, Hajiaghayi & Lee'08]
  - Constant-factor approximation NP-hard for any constant, assuming SSE-conjecture [Wu, Austrin, Pitassi & Liu'14]
- FPT-approximation:
  - $2^{\mathcal{O}(k)}n^2$ time 4-approximation [Robertson & Seymour'86]
  - $k^{\mathcal{O}(k)}n \log^2 n$ [Matoušek and Thomas'91, Lagergren'91] and $k^{\mathcal{O}(k)}n \log n$ time [Reed'92] approximations

Approximating treewidth

- Polynomial-time approximation:
  - $\mathcal{O}(\sqrt{\log(k)})$-approximation [Feige, Hajiaghayi & Lee'08]
  - Constant-factor approximation NP-hard for any constant, assuming SSE-conjecture [Wu, Austrin, Pitassi & Liu'14]
- FPT-approximation:
  - $2^{\mathcal{O}(k)}n^2$ time 4-approximation [Robertson & Seymour'86]
  - $k^{\mathcal{O}(k)}n\log^2 n$ [Matoušek and Thomas'91, Lagergren'91] and $k^{\mathcal{O}(k)}n\log n$ time [Reed'92] approximations
  - $2^{\mathcal{O}(k)}n$ time 5-approximation [Bodlaender, Drange, Dregi, Fomin, Lokshtanov & Pilipczuk'16]

Approximating treewidth

- Polynomial-time approximation:
  - $\mathcal{O}(\sqrt{\log(k)})$-approximation [Feige, Hajiaghayi & Lee'08]
  - Constant-factor approximation NP-hard for any constant, assuming SSE-conjecture [Wu, Austrin, Pitassi & Liu'14]
- FPT-approximation:
  - $2^{\mathcal{O}(k)} n^2$ time 4-approximation [Robertson & Seymour'86]
  - $k^{\mathcal{O}(k)} n \log^2 n$ [Matoušek and Thomas'91, Lagergren'91] and $k^{\mathcal{O}(k)} n \log n$ time [Reed'92] approximations
  - $2^{\mathcal{O}(k)} n$ time 5-approximation [Bodlaender, Drange, Dregi, Fomin, Lokshtanov & Pilipczuk'16]
  - $2^{\mathcal{O}(k)} n$ time 2-approximation [K. '21]

Approximating treewidth

- Polynomial-time approximation:
  - $\mathcal{O}(\sqrt{\log(k)})$-approximation [Feige, Hajiaghayi & Lee'08]
  - Constant-factor approximation NP-hard for any constant, assuming SSE-conjecture [Wu, Austrin, Pitassi & Liu'14]
- FPT-approximation:
  - $2^{\mathcal{O}(k)} n^2$ time 4-approximation [Robertson & Seymour'86]
  - $k^{\mathcal{O}(k)} n \log^2 n$ [Matoušek and Thomas'91, Lagergren'91] and $k^{\mathcal{O}(k)} n \log n$ time [Reed'92] approximations
  - $2^{\mathcal{O}(k)} n$ time 5-approximation [Bodlaender, Drange, Dregi, Fomin, Lokshtanov & Pilipczuk'16]
  - $2^{\mathcal{O}(k)} n$ time 2-approximation [K. '21]

Theorem (This work)

There is a $k^{\mathcal{O}(k/\varepsilon)} n^4$ time $(1+\varepsilon)$-approximation algorithm for treewidth.

Outline

Outline

1. How to improve a tree decomposition

Suffices to solve the *Subset treewidth* problem

Outline

1. How to improve a tree decomposition

Suffices to solve the *Subset treewidth* problem

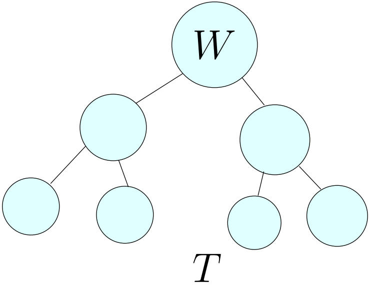
2. Solving the subset treewidth problem

Algorithms for subset treewidth that then imply algorithms for treewidth

How to improve a tree decomposition

Setting

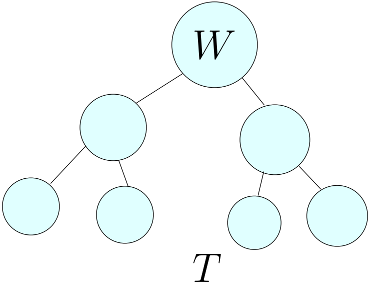
Suppose we have a tree decomposition T whose largest bag is W



Setting

Suppose we have a tree decomposition T whose largest bag is W

Goal:

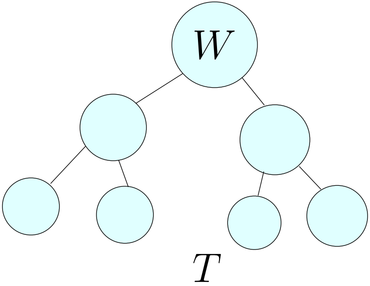


Setting

Suppose we have a tree decomposition T whose largest bag is W

Goal:

1. either decrease the number of bags of size $|W|$ while not increasing the width of T, or

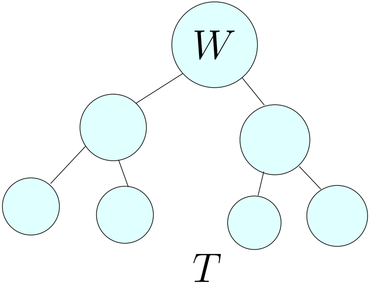


Setting

Suppose we have a tree decomposition T whose largest bag is W

Goal:

1. either decrease the number of bags of size $|W|$ while not increasing the width of T, or
2. conclude that T is (approximately) optimal



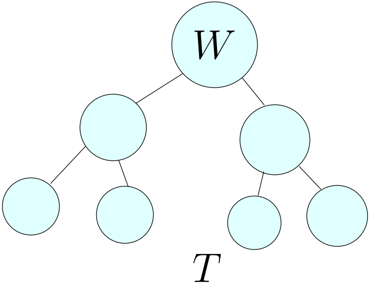
Setting

Suppose we have a tree decomposition T whose largest bag is W

Goal:

1. either decrease the number of bags of size $|W|$ while not increasing the width of T, or
2. conclude that T is (approximately) optimal

Repeat for $\mathcal{O}(\text{tw}(G) \cdot n)$ iterations to get an (approximately) optimal tree decomposition



Setting

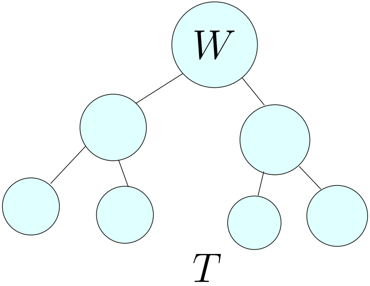
Suppose we have a tree decomposition T whose largest bag is W

Goal:

1. either decrease the number of bags of size $|W|$ while not increasing the width of T, or
2. conclude that T is (approximately) optimal

Repeat for $\mathcal{O}(\mathrm{tw}(G) \cdot n)$ iterations to get an (approximately) optimal tree decomposition

(assume to start with width $\mathcal{O}(\mathrm{tw}(G))$ decomposition)



Improving a tree decomposition

Let W be a largest bag of T

Improving a tree decomposition

Let W be a largest bag of T

Want to find:

Improving a tree decomposition

Let W be a largest bag of T

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and

Improving a tree decomposition

Let W be a largest bag of T

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Improving a tree decomposition

Let W be a largest bag of T SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\texttt{torso}(X)$ of width $\leq |W| - 2$

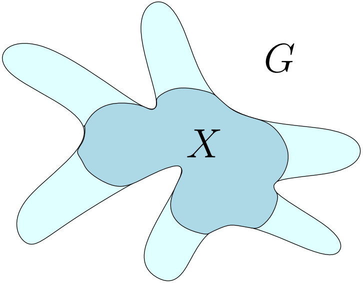
Improving a tree decomposition

Let W be a largest bag of T — SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Torso?



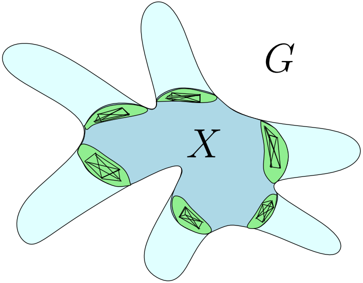
Improving a tree decomposition

Let W be a largest bag of T SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Torso?



- Make neighborhoods of components of $G \setminus X$ into cliques

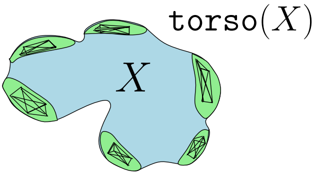
Improving a tree decomposition

Let W be a largest bag of T — SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\mathtt{torso}(X)$ of width $\leq |W| - 2$

Torso?



- Make neighborhoods of components of $G \setminus X$ into cliques
- Delete $V(G) \setminus X$

Improving a tree decomposition

Let W be a largest bag of T SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Observations:

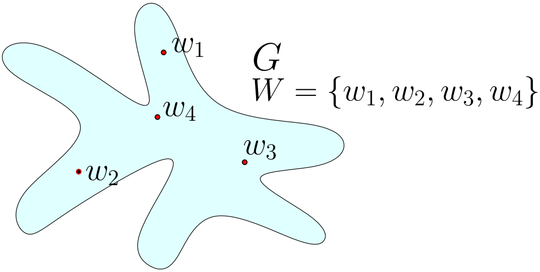
Improving a tree decomposition

Let W be a largest bag of T — SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Observations:



Improving a tree decomposition

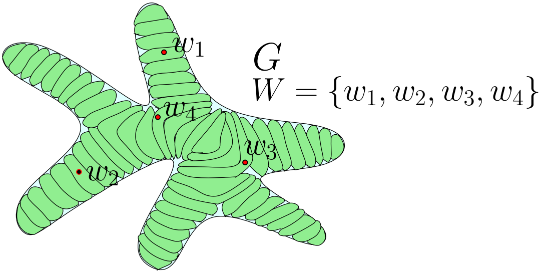
Let W be a largest bag of T — SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Observations:

- If T is not optimal, then such X exists by taking $X = V(G)$



Improving a tree decomposition

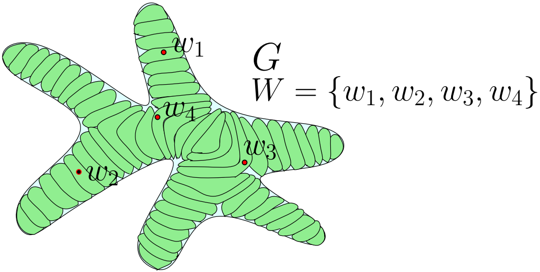
Let W be a largest bag of T SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Observations:

- If T is not optimal, then such X exists by taking $X = V(G)$
- Freedom to choose $X \subset V(G)$



Improving a tree decomposition

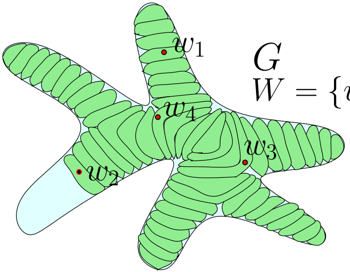
Let W be a largest bag of T — SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Observations:

- If T is not optimal, then such X exists by taking $X = V(G)$
- Freedom to choose $X \subset V(G)$



G

$W = \{w_1, w_2, w_3, w_4\}$

Improving a tree decomposition

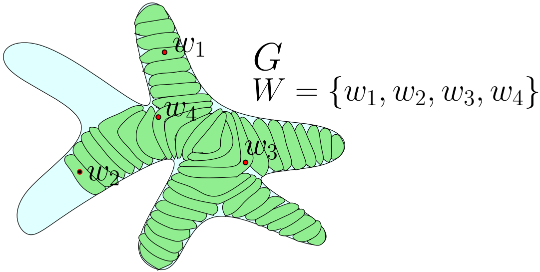
Let W be a largest bag of T SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Observations:

- If T is not optimal, then such X exists by taking $X = V(G)$
- Freedom to choose $X \subset V(G)$



Improving a tree decomposition

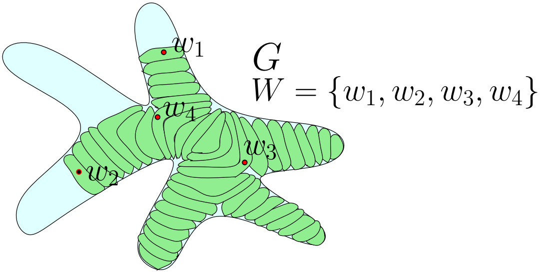
Let W be a largest bag of T — SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Observations:

- If T is not optimal, then such X exists by taking $X = V(G)$
- Freedom to choose $X \subset V(G)$



Improving a tree decomposition

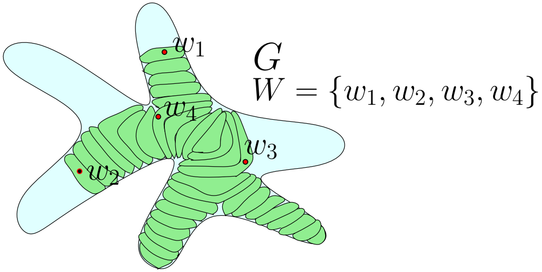
Let W be a largest bag of T — SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\mathtt{torso}(X)$ of width $\leq |W| - 2$

Observations:

- If T is not optimal, then such X exists by taking $X = V(G)$
- Freedom to choose $X \subset V(G)$



Improving a tree decomposition

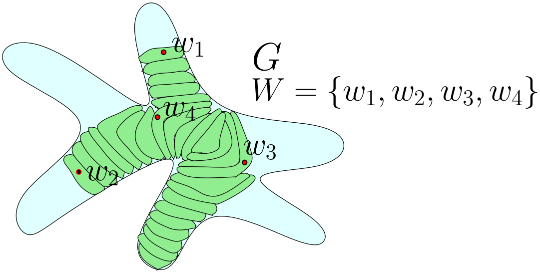
Let W be a largest bag of T — SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Observations:

- If T is not optimal, then such X exists by taking $X = V(G)$
- Freedom to choose $X \subset V(G)$



Improving a tree decomposition

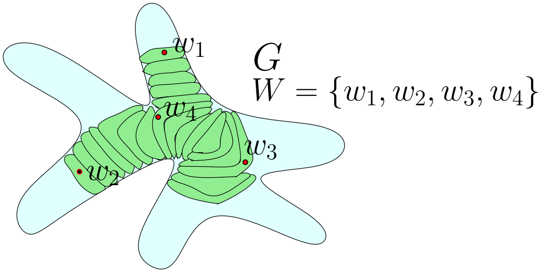
Let W be a largest bag of T — SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Observations:

- If T is not optimal, then such X exists by taking $X = V(G)$
- Freedom to choose $X \subset V(G)$



Improving a tree decomposition

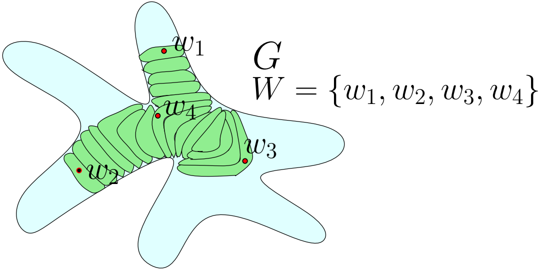
Let W be a largest bag of T SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Observations:

- If T is not optimal, then such X exists by taking $X = V(G)$
- Freedom to choose $X \subset V(G)$



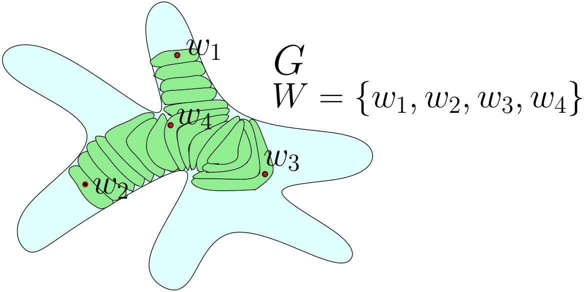
Improving a tree decomposition

Let W be a largest bag of T SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\mathtt{torso}(X)$ of width $\leq |W| - 2$

Big-leaf formulation:



Improving a tree decomposition

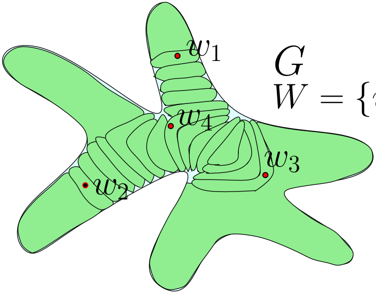
Let W be a largest bag of T SUBSET TREEWIDTH

Want to find:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Big-leaf formulation:

- Find a tree decomposition of G whose internal bags have size $\leq |W| - 1$ and cover W, but leaf bags can be arbitrarily large



G

$W = \{w_1, w_2, w_3, w_4\}$

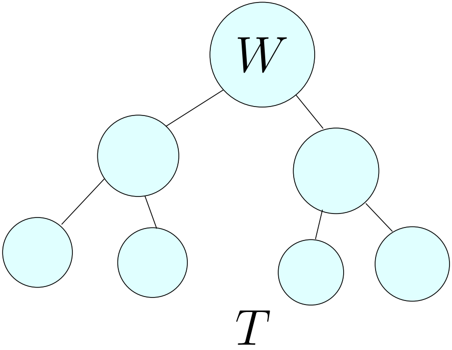
Improving a tree decomposition

Let W be a largest bag of T SUBSET TREEWIDTH

Have:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition T_X of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Improving T:



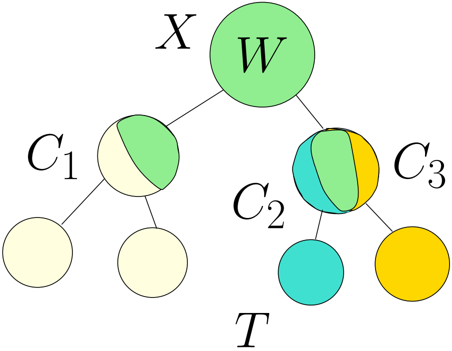
Improving a tree decomposition

Let W be a largest bag of T SUBSET TREEWIDTH

Have:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition T_X of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Improving T:



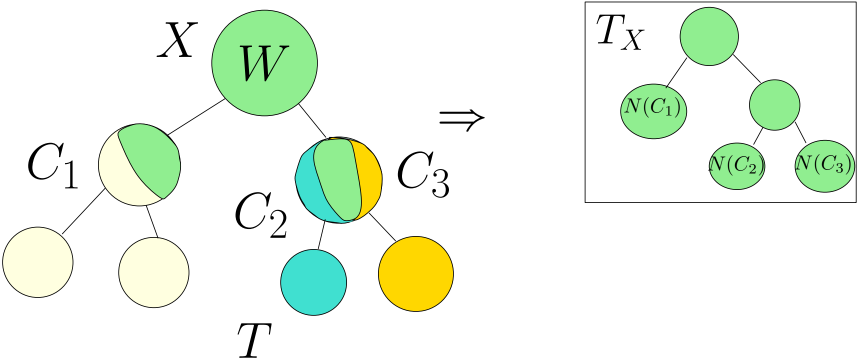
Improving a tree decomposition

Let W be a largest bag of T SUBSET TREEWIDTH

Have:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition T_X of $\texttt{torso}(X)$ of width $\leq |W| - 2$

Improving T:



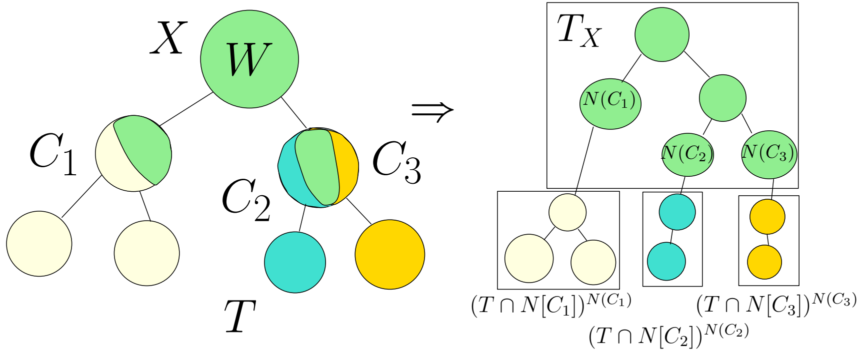
Improving a tree decomposition

Let W be a largest bag of T — SUBSET TREEWIDTH

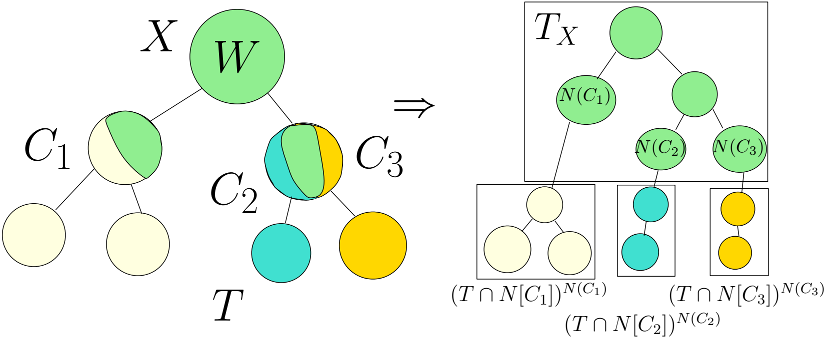
Have:

- a set X with $W \subseteq X \subseteq V(G)$, and
- a tree decomposition T_X of $\texttt{torso}(X)$ of width $\leq |W| - 2$

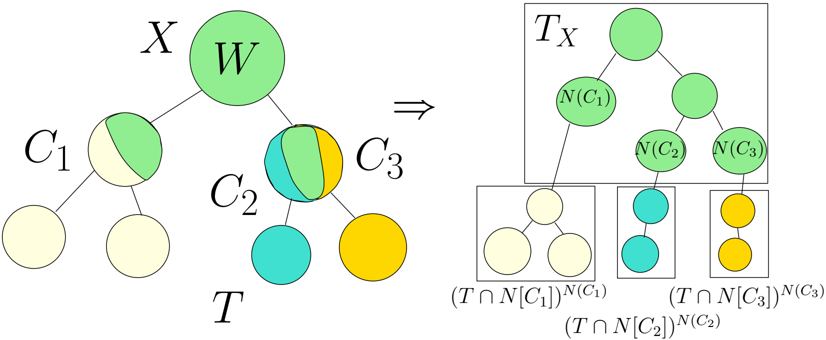
Improving T:



Does T improve?

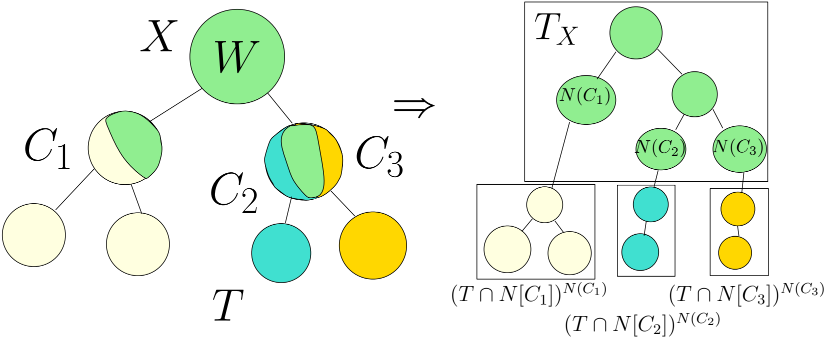


Does T improve?



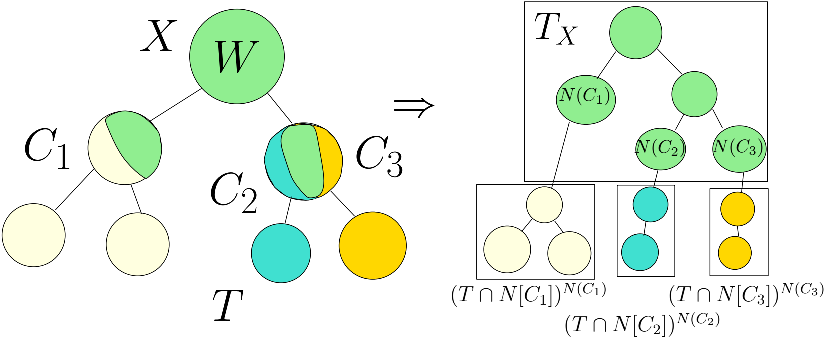
- Want: The copy of a bag in $(T \cap N[C_i])^{N(C_i)}$ is not larger than the original bag

Does T improve?



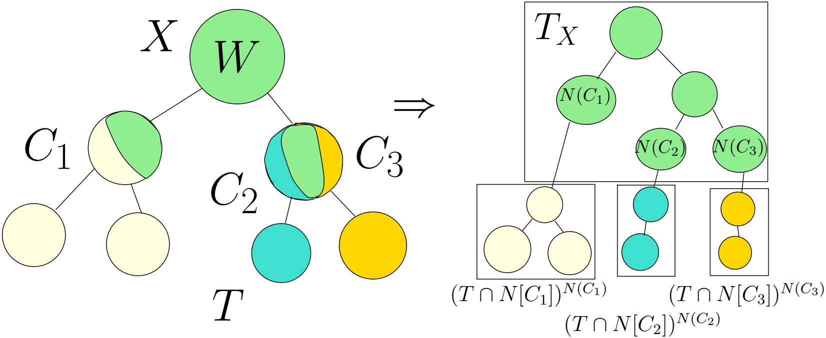
- Want: The copy of a bag in $(T \cap N[C_i])^{N(C_i)}$ is not larger than the original bag
- This holds if T_X is preprocessed so that its every bag is linked into W

Does T improve?



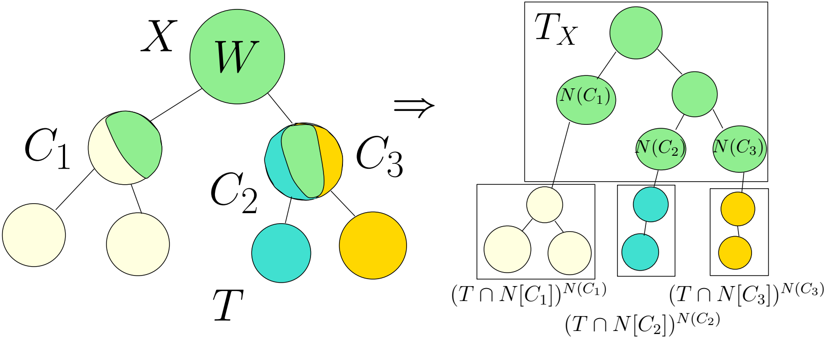
- Want: The copy of a bag in $(T \cap N[C_i])^{N(C_i)}$ is not larger than the original bag
- This holds if T_X is preprocessed so that its every bag is linked into W
  - $k^{\mathcal{O}(1)} n^4$ time here

Does T improve?



- Want: The copy of a bag in $(T \cap N[C_i])^{N(C_i)}$ is not larger than the original bag
- This holds if T_X is preprocessed so that its every bag is linked into W
  - $k^{\mathcal{O}(1)} n^4$ time here
- Proofs by Bellenbaum-Diestel type arguments

Does T improve?



- Want: The copy of a bag in $(T \cap N[C_i])^{N(C_i)}$ is not larger than the original bag
- This holds if T_X is preprocessed so that its every bag is linked into W
  - $k^{\mathcal{O}(1)} n^4$ time here
- Proofs by Bellenbaum-Diestel type arguments
- (actually need a bit stronger condition than linkedness for improvement)

Subset treewidth for exact algorithms

Subset treewidth for exact algorithms

SUBSET TREEWIDTH

Input: Graph G, integer k, set of vertices $W \subseteq V(G)$ with $|W| = k + 2$

Output: Set $X \subseteq V(G)$ with $W \subseteq X$ and tree decomposition of $\texttt{torso}(X)$ of width $\leq k$ or that the treewidth of G is $> k$

Subset treewidth for exact algorithms

SUBSET TREEWIDTH

Input: Graph G, integer k, set of vertices $W \subseteq V(G)$ with $|W| = k + 2$

Output: Set $X \subseteq V(G)$ with $W \subseteq X$ and tree decomposition of $\texttt{torso}(X)$ of width $\leq k$ or that the treewidth of G is $> k$

Theorem

If there is an $f(k) \cdot n^{\mathcal{O}(1)}$ time algorithm for subset treewidth, then there is an $f(k) \cdot n^{\mathcal{O}(1)}$ time algorithm for treewidth with the same function f.

Subset treewidth for exact algorithms

SUBSET TREEWIDTH

Input: Graph G, integer k, set of vertices $W \subseteq V(G)$ with $|W| = k + 2$

Output: Set $X \subseteq V(G)$ with $W \subseteq X$ and tree decomposition of $\mathtt{torso}(X)$ of width $\leq k$ or that the treewidth of G is $> k$

Theorem

If there is an $f(k) \cdot n^{\mathcal{O}(1)}$ time algorithm for subset treewidth, then there is an $f(k) \cdot n^{\mathcal{O}(1)}$ time algorithm for treewidth with the same function f.

(actually *if and only if*)

Subset treewidth for exact algorithms

SUBSET TREEWIDTH

Input: Graph G, integer k, set of vertices $W \subseteq V(G)$ with $|W| = k + 2$

Output: Set $X \subseteq V(G)$ with $W \subseteq X$ and tree decomposition of $\mathtt{torso}(X)$ of width $\leq k$ or that the treewidth of G is $> k$

Theorem

If there is an $f(k) \cdot n^{\mathcal{O}(1)}$ time algorithm for subset treewidth, then there is an $f(k) \cdot n^{\mathcal{O}(1)}$ time algorithm for treewidth with the same function f.

(actually *if and only if*)

$2^{\mathcal{O}(k^2)} n^2$ time algorithm for subset treewidth $\rightarrow$ $2^{\mathcal{O}(k^2)} n^4$ time algorithm for treewidth

Subset treewidth for approximation schemes

Subset treewidth for approximation schemes

PARTITIONED SUBSET TREEWIDTH

Input: Graph G, integer k, set of vertices $W \subseteq V(G)$ with $|W| = k+2$ that is partitioned into t cliques $W_1, \ldots, W_t$

Output: Set $X \subseteq V(G)$ with $W \subseteq X$ and tree decomposition of $\texttt{torso}(X)$ of width $\leq k$ or that the treewidth of G is $> k$

Subset treewidth for approximation schemes

PARTITIONED SUBSET TREEWIDTH

Input: Graph G, integer k, set of vertices $W \subseteq V(G)$ with $|W| = k+2$ that is partitioned into t cliques $W_1, \ldots, W_t$

Output: Set $X \subseteq V(G)$ with $W \subseteq X$ and tree decomposition of $\mathtt{torso}(X)$ of width $\leq k$ or that the treewidth of G is $> k$

Theorem

If there is an $f(k,t) \cdot n^{\mathcal{O}(1)}$ time algorithm for partitioned subset treewidth, then there is a $f(\mathcal{O}(k), \mathcal{O}(1/\varepsilon)) \cdot k^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ time $(1+\varepsilon)$-approximation algorithm for treewidth with the same function f.

Subset treewidth for approximation schemes

PARTITIONED SUBSET TREEWIDTH

Input: Graph G, integer k, set of vertices $W \subseteq V(G)$ with $|W| = k+2$ that is partitioned into t cliques $W_1, \ldots, W_t$

Output: Set $X \subseteq V(G)$ with $W \subseteq X$ and tree decomposition of $\texttt{torso}(X)$ of width $\leq k$ or that the treewidth of G is $> k$

Theorem

If there is an $f(k,t) \cdot n^{\mathcal{O}(1)}$ time algorithm for partitioned subset treewidth, then there is a $f(\mathcal{O}(k), \mathcal{O}(1/\varepsilon)) \cdot k^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ time $(1+\varepsilon)$-approximation algorithm for treewidth with the same function f.

$k^{\mathcal{O}(kt)} n^2$ time algorithm for partitioned subset treewidth $\rightarrow k^{\mathcal{O}(k/\varepsilon)} n^4$ time $(1+\varepsilon)$-approximation algorithm for treewidth

Solving the subset treewidth problem

Solving the subset treewidth problem

Goal: Sketch $k^{\mathcal{O}(kt)} n^{\mathcal{O}(1)}$ time algorithm for partitioned subset treewidth

Solving the subset treewidth problem

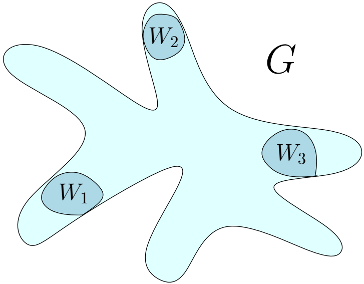
Goal: Sketch $k^{\mathcal{O}(kt)} n^{\mathcal{O}(1)}$ time algorithm for partitioned subset treewidth

(this is also a $k^{\mathcal{O}(k^2)} n^{\mathcal{O}(1)}$ time algorithm for subset treewidth)

Solving subset treewidth

Setting:

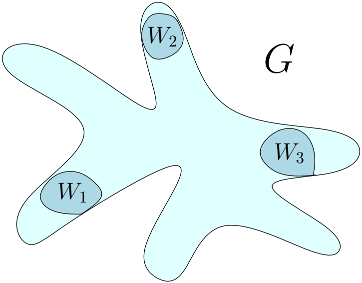
- Input: Graph G, t terminal cliques $W_1, \ldots, W_t$, and an integer k



Solving subset treewidth

Setting:

- Input: Graph G, t terminal cliques $W_1, \ldots, W_t$, and an integer k
- Goal: Find $X \supseteq \bigcup_{i=1}^{t} W_i$ and a tree decomposition of $\mathtt{torso}(X)$ of width $\leq k$

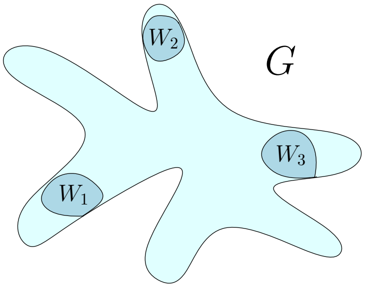


Solving subset treewidth

Setting:

- Input: Graph G, t terminal cliques $W_1, \ldots, W_t$, and an integer k
- Goal: Find $X \supseteq \bigcup_{i=1}^{t} W_i$ and a tree decomposition of $\texttt{torso}(X)$ of width $\leq k$

Reduction rule:



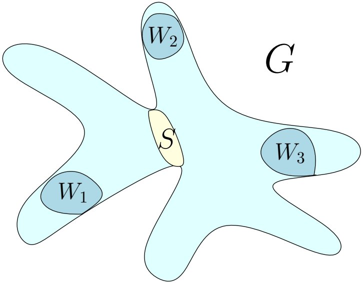
Solving subset treewidth

Setting:

- Input: Graph G, t terminal cliques $W_1, \ldots, W_t$, and an integer k
- Goal: Find $X \supseteq \bigcup_{i=1}^{t} W_i$ and a tree decomposition of $\texttt{torso}(X)$ of width $\leq k$

Reduction rule:

Let S be a non-trivial minimum size (W_i, W_j)-separator



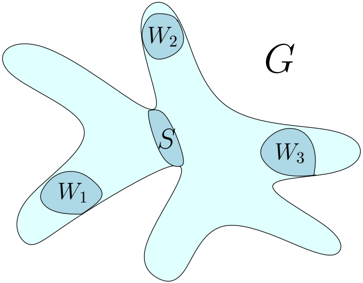
Solving subset treewidth

Setting:

- Input: Graph G, t terminal cliques $W_1, \dots, W_t$, and an integer k
- Goal: Find $X \supseteq \bigcup_{i=1}^{t} W_i$ and a tree decomposition of $\texttt{torso}(X)$ of width $\leq k$

Reduction rule:

Let S be a non-trivial minimum size (W_i, W_j)-separator
Make S into a terminal clique and solve both sides independently



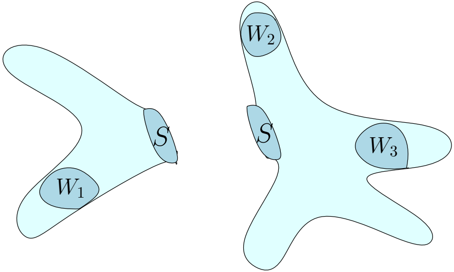
Solving subset treewidth

Setting:

- Input: Graph G, t terminal cliques $W_1, \ldots, W_t$, and an integer k
- Goal: Find $X \supseteq \bigcup_{i=1}^{t} W_i$ and a tree decomposition of $\texttt{torso}(X)$ of width $\leq k$

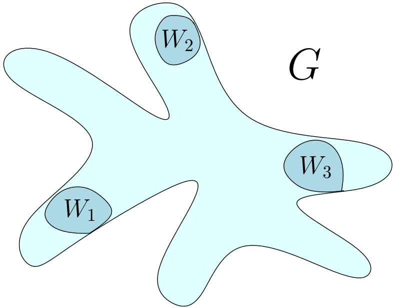
Reduction rule:

Let S be a non-trivial minimum size (W_i, W_j)-separator
Make S into a terminal clique and solve both sides independently



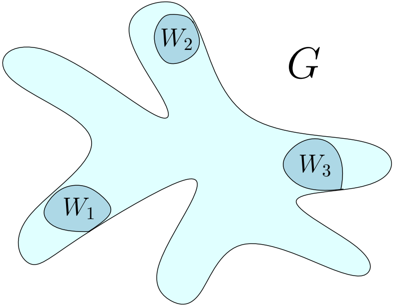
Branching for partitioned subset treewidth

- Now terminal cliques *strongly linked* into each other



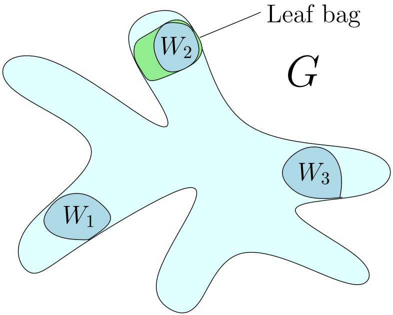
Branching for partitioned subset treewidth

- Now terminal cliques *strongly linked* into each other
- Goal: To make progress, increase the size/flow of some terminal clique



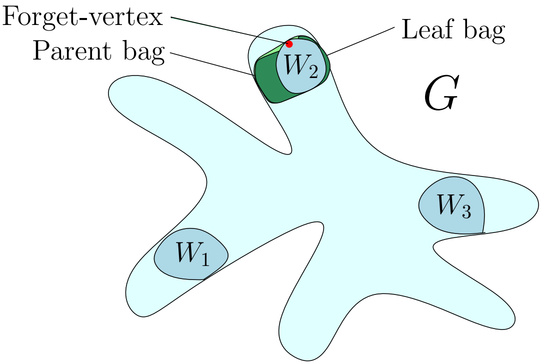
Branching for partitioned subset treewidth

- Now terminal cliques *strongly linked* into each other
- Goal: To make progress, increase the size/flow of some terminal clique



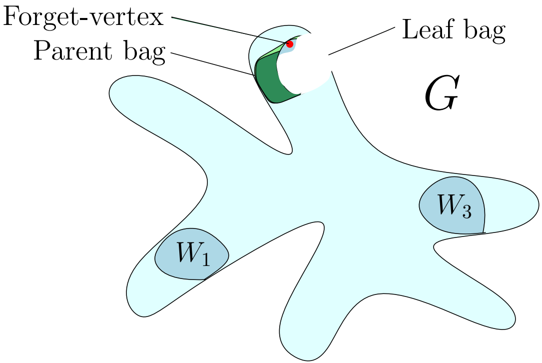
Branching for partitioned subset treewidth

- Now terminal cliques *strongly linked* into each other
- Goal: To make progress, increase the size/flow of some terminal clique



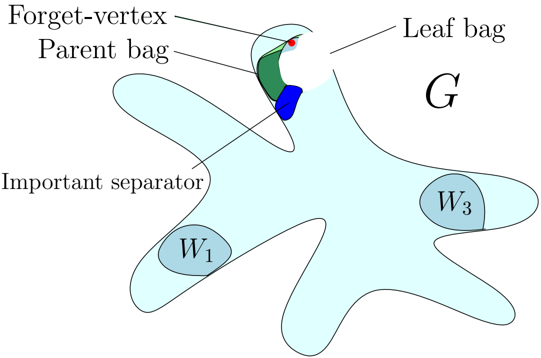
Branching for partitioned subset treewidth

- Now terminal cliques *strongly linked* into each other
- Goal: To make progress, increase the size/flow of some terminal clique



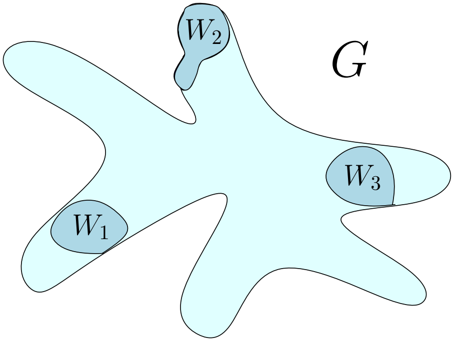
Branching for partitioned subset treewidth

- Now terminal cliques *strongly linked* into each other
- Goal: To make progress, increase the size/flow of some terminal clique
- Increase W_2 by guessing an important separator

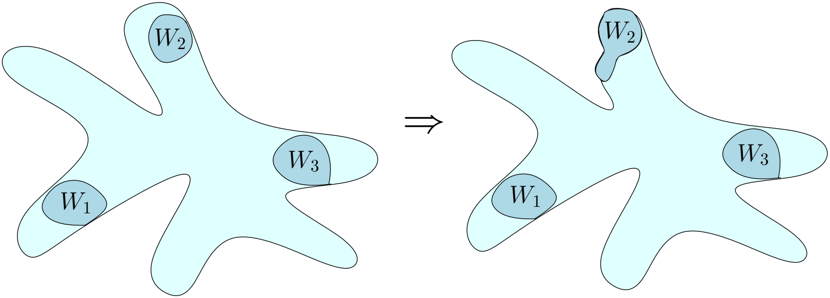


Branching for partitioned subset treewidth

- Now terminal cliques *strongly linked* into each other
- Goal: To make progress, increase the size/flow of some terminal clique
- Increase W_2 by guessing an important separator

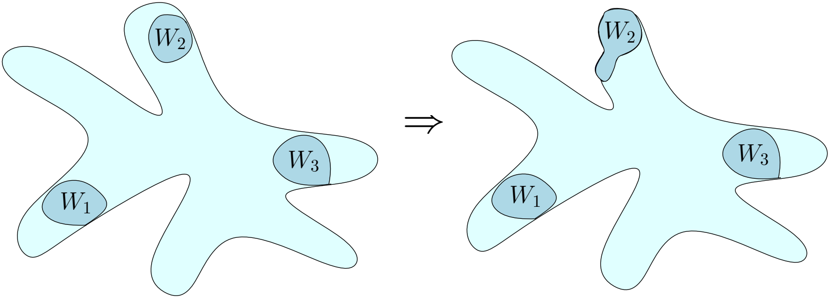


Analysis of branching



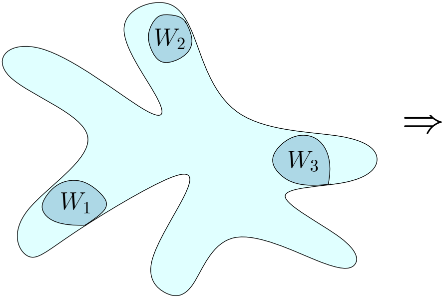
- Increased the size/flow of a leaf terminal clique by guessing a forget-vertex and an important separator

Analysis of branching



- Increased the size/flow of a leaf terminal clique by guessing a forget-vertex and an important separator
- Sum of sizes/flows of terminal cliques at most $(k+1)t$, so branching depth at most kt

Analysis of branching



- Increased the size/flow of a leaf terminal clique by guessing a forget-vertex and an important separator
- Sum of sizes/flows of terminal cliques at most $(k+1)t$, so branching depth at most kt
- To get $k^{\mathcal{O}(kt)} n^{\mathcal{O}(1)}$ time, need also an important separator hitting set lemma

Conclusion

Open questions:

Conclusion

Open questions:

- Is there $2^{\mathcal{O}(k^{1.999})} n^{\mathcal{O}(1)}$ time algorithm for subset treewidth?

Conclusion

Open questions:

- Is there $2^{\mathcal{O}(k^{1.999})} n^{\mathcal{O}(1)}$ time algorithm for subset treewidth?
- When $t = \mathcal{O}(1)$, is there $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ time algorithm for partitioned subset treewidth?

Conclusion

Open questions:

- Is there $2^{\mathcal{O}(k^{1.999})} n^{\mathcal{O}(1)}$ time algorithm for subset treewidth?
- When $t = \mathcal{O}(1)$, is there $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ time algorithm for partitioned subset treewidth?
- How much can the n^4 factor be optimized?

Thank you!

Thank you!